ELEMENTS OF PLANE TRIGONOMETRY

(For Pre-University & Higher Secondary Classes)

By

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SYLLABUS FOR THE PRE-UNIVERSITY EXAMINATION.

Relations between Trigonometrical Ratios of angles differing by multiple of right angles, addition and subtraction formulae; Trigonometrical ratios of multiples and sub-multiples of angles. General solution of simple Trigonometrical equations, the relations between the sides and the angles of a triangle. Logarithms, solution of triangles and simple cases of heights and distances, radii of the circumscribed, inscribed and escribed circles of triangles; areas of a triangle, regular polygon and circle; graphs of simple trigonometrical functions.

PREFACE

The following work is meant to be an elementary text-book on Plane Trigonometry, suitable for the Pre-university and Higher Secondary Classes of the Jammu and Kashmir University. An effort has been made to make treatment of the the subject lucid and concise. A large number of examples, selected from Question Papers of various University Examinations, have been solved to illustrate the application of formulae.

The topics of *solution of triangles* and *heights* and *distances*, which deal with the manifold practical applications of the subject, have been treated at greater length.

Question papers set at the Intermediate Examination of the J. & K University and the Higher Secondary Examination of the Delhi and J. & K. Universities have been printed at the end.

Acknowledgement is hereby made to all the authors consulted in the preparation of the book.

K. L. Varma.

JAMMU,
January 1, 1962.

IMPORTANT FORMULAE
AT A GLANCE

I. 1 rt. angle $=90°$, $1°=60'$, $1'=60''$.

1 rt. angle $=100^g$, $1^g=100'$, $1'=100''$

Circumference of a circle (of radius r) $=2\pi r$

$\pi=\frac{22}{7}$ (nearly). More accurately $\pi=\frac{355}{113}=3\cdot14159\ldots$

π radians $=2$ rt. angles $=180°=200^g$.

No. of *radians* in an angle subtended by an arc at the centre of a circle $=\dfrac{\text{arc}}{\text{radius}}$ $\left[\text{or } \theta=\dfrac{l}{r}\right]$

II. $\sin\theta=\dfrac{MP}{OP}\left(=\dfrac{\text{opp. side}}{\text{hypot.}}\right)$

$\cos\theta=\dfrac{OM}{OP}\left(=\dfrac{\text{adj. side}}{\text{hypot.}}\right)$

$\tan\theta=\dfrac{MP}{OM}\left(=\dfrac{\text{opp. side}}{\text{adj. side}}\right)$

$\cot\theta=\dfrac{OM}{MP}$, $\sec\theta=\dfrac{OP}{OM}$, $\operatorname{cosec}\theta=\dfrac{OP}{MP}$.

$\operatorname{cosec}\theta=\dfrac{1}{\sin\theta}$, $\sec\theta=\dfrac{1}{\cos\theta}$,

$\cot\theta=\dfrac{1}{\tan\theta}$

$\tan\theta=\dfrac{\sin\theta}{\cos\theta}$, $\cot\theta=\dfrac{\cos\theta}{\sin\theta}$

$\sin^2\theta+\cos^2\theta=1$.

$\sec^2\theta=1+\tan^2\theta$.

$\operatorname{cosec}^2\theta=1+\cot^2\theta$.

III.

θ	$0°$	$30°$	$45°$	$60°$	$90°$
$\sin\theta$	$\sqrt{\frac{0}{4}}$	$\sqrt{\frac{1}{4}}$	$\sqrt{\frac{2}{4}}$	$\sqrt{\frac{3}{4}}$	$\sqrt{\frac{4}{4}}$
$\cos\theta$	$\sqrt{\frac{4}{4}}$	$\sqrt{\frac{3}{4}}$	$\sqrt{\frac{2}{4}}$	$\sqrt{\frac{1}{4}}$	$\sqrt{\frac{0}{4}}$

IV. $\sin(-\theta)=-\sin\theta$, $\cos(-\theta)=\cos\theta$, $\tan(-\theta)=-\tan\theta$

$\sin(90°-\theta)=\cos\theta$	$\sin(90°+\theta)=\cos\theta$
$\cos(90°-\theta)=\sin\theta$	$\cos(90°+\theta)=-\sin\theta$
$\tan(90°-\theta)=\cot\theta$	$\tan(90°+\theta)=-\cot\theta.$

V.

$\sin(180°-\theta)=\sin\theta$	$\sin(180°+\theta)=-\sin\theta.$
$\cos(180°-\theta)=-\cos\theta$	$\cos(180°+\theta)=-\cos\theta.$
$\tan(180°-\theta)=-\tan\theta$	$\tan(180°+\theta)=\tan\theta.$

VI. $\sin(A+B)=\sin A\cos B+\cos A\sin B.$

$\cos(A+B)=\cos A\cos B-\sin A\sin B.$

$$\tan(A+B)=\frac{\tan A+\tan B}{1-\tan A\tan B}.$$

$\sin(A-B)=\sin A\cos B-\cos A\sin B.$

$\cos(A-B)=\cos A\cos B+\sin A\sin B.$

$$\tan(A-B)=\frac{\tan A-\tan B}{1+\tan A\tan B}.$$

$\sin(A+B)\sin(A-B)=\sin^2A-\sin^2B.$

$\cos(A+B)\cos(A-B)=\cos^2A-\sin^2B.$

$$\tan(45°+A)=\frac{1+\tan A}{1-\tan A},\quad \tan(45°-A)=\frac{1-\tan A}{1+\tan A}.$$

$$\tan(A+B+C)=\frac{\tan A+\tan B+\tan C-\tan A\tan B\tan C}{1-\tan B\tan C-\tan C\tan A-\tan A\tan B}$$

VII. $2\sin A\cos B=\sin(A+B)+\sin(A-B)$...(1)

$2\cos A\sin B=\sin(A+B)-\sin(A-B)$...(2)

$2\cos A\cos B=\cos(A+B)+\cos(A-B)$...(3)

$$2 \sin A \sin B = \cos (A-B) - \cos (A+B) \ldots (4)$$

$$\sin C + \sin D = 2 \sin \frac{C+D}{2} \cos \frac{G-D}{2} \ldots (1)$$

$$\sin C - \sin D = 2 \cos \frac{C+D}{2} \cos \frac{C-D}{2} \ldots (2)$$

$$\cos C + \cos D = 2 \cos \frac{C+D}{2} \cos \frac{C-D}{2} \ldots (3)$$

$$\cos C - \cos D = 2 \sin \frac{C+D}{2} \sin \frac{D-C}{2} \ldots (4)$$

VIII. $\sin 2A = 2 \sin A \cos A$.

$$\cos 2A = \cos^2 A - \sin^2 A \qquad (1)$$

$$= 1 - 2 \sin^2 A. \qquad (2)$$

$$= 2 \cos^2 A - 1. \qquad (3)$$

$$\tan 2A = \frac{2 \tan A}{1 - \tan^2 A}$$

$$\sin 2A = \frac{2 \tan A}{1 + \tan^2 A}, \quad \cos 2A = \frac{1 - \tan^2 A}{1 + \tan^2 A}$$

$$\sin 3A = 3 \sin A - 4 \sin^3 A.$$

$$\cos 3A = 4 \cos^3 A - 3 \cos A.$$

$$\tan 3A = \frac{3 \tan A - \tan^3 A}{1 - 3 \tan^2 A}.$$

$$\sin 18^\circ = \frac{\sqrt{5}-1}{4}, \quad \cos 36^\circ = \frac{\sqrt{5}+1}{4}.$$

$$\sin \frac{A}{2} = \pm \sqrt{\frac{1 - \cos A}{2}}, \quad \cos \frac{A}{2} = \pm \sqrt{\frac{1 + \cos A}{2}}.$$

$$\tan \frac{A}{2} = \pm \sqrt{\frac{1 - \cos A}{1 + \cos A}}$$

$$\sin \frac{A}{2} + \cos \frac{A}{2} = \pm \sqrt{1 + \sin A} \qquad \ldots (1)$$

$$\sin\frac{A}{3}-\cos\frac{A}{2}=\pm\sqrt{1-\sin A} \qquad \ldots(2)$$

$\sin\frac{A}{2}+\cos\frac{A}{2}$ has the same sign as $\sin\left(\frac{A}{2}+45°\right)$

and $\sin\frac{A}{2}-\cos\frac{A}{2}$ „ „ „ $\sin\left(\frac{A}{2}-45°\right)$

IX. If $\sin\theta=0$, then $\theta=n\pi$
If $\cos\theta=0$, then $\theta=(2n+1)\frac{\pi}{2}$
} where n is zero, or any integer, positive or negative.

If $\sin\theta=\sin\alpha$, then $\theta=n\pi+(-1)^n\alpha$
If $\cos\theta=\cos\alpha$, then $\theta=2n\pi\pm\alpha$
If $\tan\theta=\tan\alpha$, then $\theta=n\pi+\alpha$
} where n is zero, or any integer, positive or negative.

X. *Sine Formula* : $\frac{a}{\sin A}=\frac{b}{\sin B}=\frac{c}{\sin C}$.

Cosine Formula : $\cos A=\frac{b^2+c^2-a^2}{2bc}$ etc.

$a^2=b^2+c^2-2bc\cos A;$..............

Projection Formula : $a=b\cos C+c$ oos B; etc.

Half-angie Fornmla : $\sin\frac{A}{2}=\sqrt{\frac{(s-b)(s-c)}{bc}}$ etc.

$\cos\frac{A}{2}=\sqrt{\frac{s(s-a)}{bc}}$, etc

$\tan\frac{A}{2}=\sqrt{\frac{s-b)(s-c)}{s(s-a)}}$, etc.

$\sin A=\frac{2}{bc}\sqrt{s(s-a)(s-b)\,s-c)}$

Napier's Analogy : $\tan\frac{B-C}{2}=\frac{b-c}{b+c}\cot\frac{A}{2}$, etc.

XI. $\log_a 1=0$, $\log_a a=1$.

$\log_a{}^{mn}=\log_a{}^m+\log_a{}^n$.

$\log_a \frac{m}{n} = \log_a m - \log_a n.$

$\log_a m^n = n \log_a m.$

$\log_a m = \log_b m \times \log_a b.$

Formula for change of base :

$$\log_b m = \frac{\log_a m}{\log_a b}.$$

XII. *Area of a triangle :*

$\triangle = \frac{1}{2} bc \sin A = \frac{1}{2} ca \sin B = \frac{1}{2} ab \sin C.$

$= \sqrt{s(s-a)(s-b)(s-c)}.$

Circumradius of a triangle :

$$R = \frac{a}{2 \sin A} = \frac{b}{2 \sin B} = \frac{c}{2 \sin C}$$

$$= \frac{abc}{4\triangle}$$

Inradius :

$$r = \frac{\triangle}{s} = (s-a) \tan \frac{A}{2} = \ldots = \ldots$$

$$= a \frac{\sin \frac{B}{2} \sin \frac{C}{2}}{\cos \frac{A}{2}}$$

Ex-radius opposite angle A :

$$r_1 = \frac{\triangle}{s-a} = s \tan \frac{A}{2}$$

$$= a \frac{\cos \frac{B}{2} \cos \frac{C}{2}}{\cos \frac{A}{2}}$$

$$r_2 = \frac{\Delta}{s-b},\quad r_3 = \frac{\Delta}{s-c}.$$

XIII. $\sin\theta < \theta < \tan\theta$, when $\theta < \frac{\pi}{2}$.

$$\operatorname*{Lt}_{\theta\to 0} \frac{\sin\theta}{\theta} = 1.$$

For a small angle θ, $\sin\theta = \theta$.

θ being the number of *radians* in the angle.

Area of a circle $= \pi r^2$.

Area of a sector of a circle $= \frac{1}{2} r^2 \theta$.

Area of a segment of a circle $= \frac{1}{2} r^2 (\theta - \sin\theta)$.

θ being the number of *radians* in the angle at the centre.

CHAPTER I

Measurement of Angles

1. Trignometery means 'measurement of a triangle'. At first it was used in 'solving a triange' (that is finding the remaining sides and angles of a triangle when some of these are known). But now its scope has widened and it is much used in surveying and navigation. There is another branch of Trignometery called *Spherical Trignometery* which finds use in Astronomy.

2. In Geometery an angle is defined as "the inclimation of two straight lines which meet." This definition does not include angles greater than two right angles. Hence we give a new definition.

Definition. An angle is the amount of revolution made by a line revolving about one of its etxremities, in a plane, from one position to another.

Let a straight line O P revolve about O from the position OX to the position OP. Then it traces out the angle XOP.

OX is called the **initial line** and OP the **revolving** line.

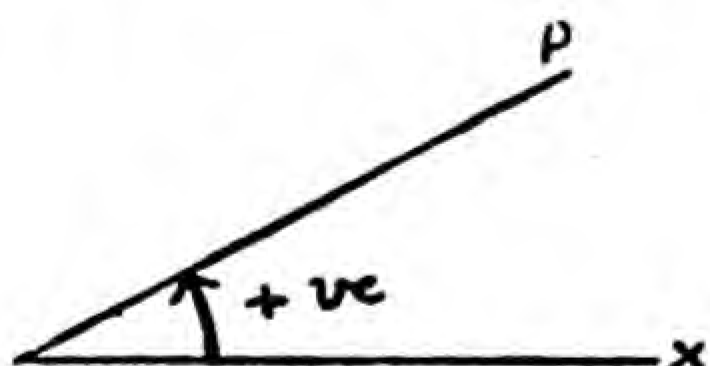


Fig 1

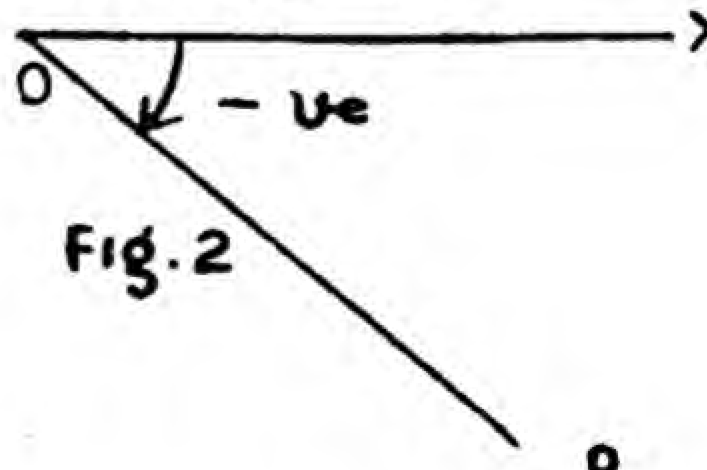
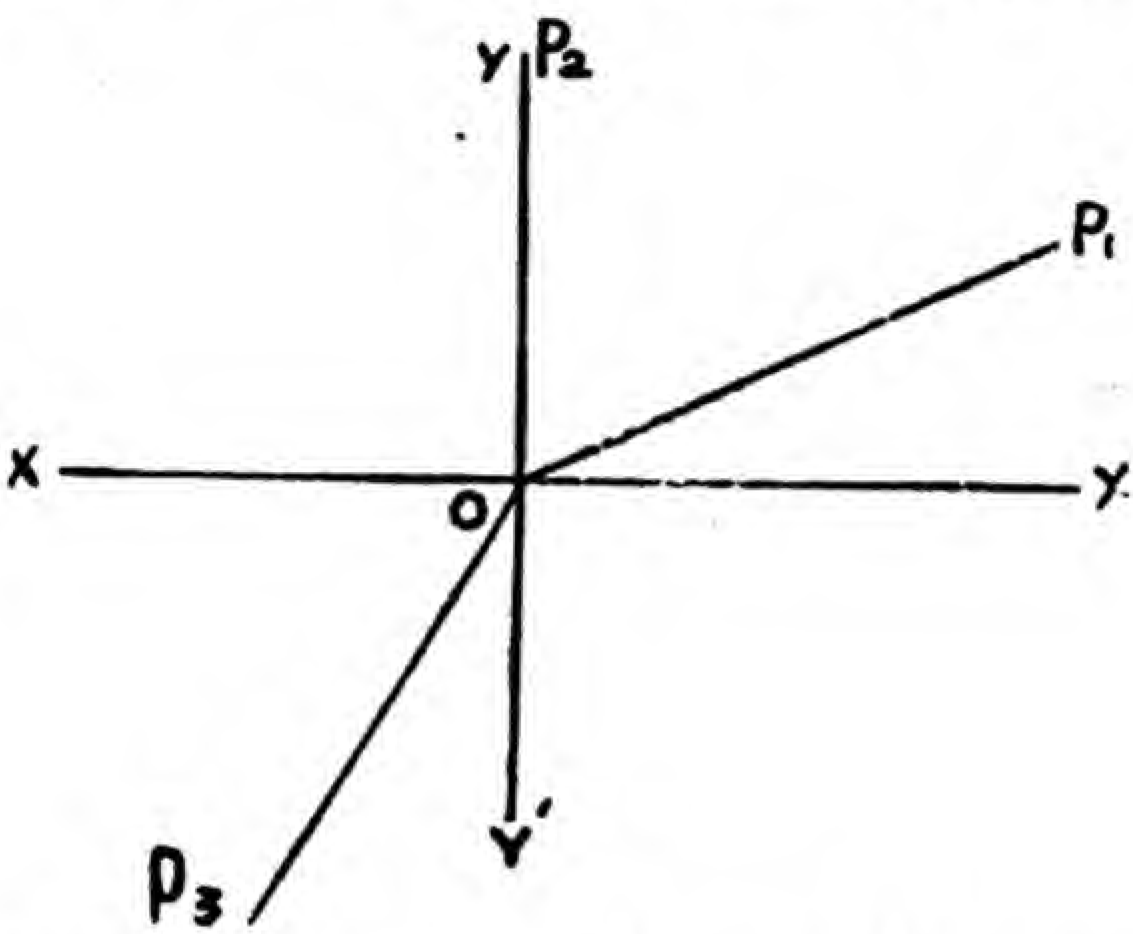


Fig. 2

If OP revolves in *anticlockwise* direction (*i. e.* in a direction opposite to the motion of the hands of a watch)

as in fig 1. the angle is **positive** and if OP revolves in clockwise direction as in fig. 2 the angle traced is **negative.** The arrow head indicates the direction in which the line OP has revolved.

3. **Angles of any magnitude** :—An angle XOP is measured by the amount of revolution of the revolving line from its initial position OX to its final position OP. Thus if OX is the initial line and YOY′ perpendicular to it, O being the origin, and the revolving ling OP starting from the initial position revolves in the anti-clockwise direction and takes the different positions OP_1, OP_2, OP_3......as shown in the figure, then the angles traced out are XOP_1, XOP_2, XOP_3......



When it coincides with OY, it has traced ∠XOY=one rt. ∠
When ,, ,, ,, OX′, ,, ,, ,, ∠XOY′=2 rt, ∠s.
,, ,, ,, ,, OY′, ,, ,, ,, ∠XOY′=3 rt. ∠s.
,, ,, ,, ,, OX after completing one revolution the angle traced out is=4 rt. ∠s.

Angle after two complete revolutions=8 rt. ∠s, and so on.

The revolving line OP may turn round to any extent and stop anywhere in its course. The angle formed is the number of complete revolutions if any, plus the visible ∠XOP.

Thus the angle may be positive or negative and of any magnitude depending upon direction and the amount of revolution.

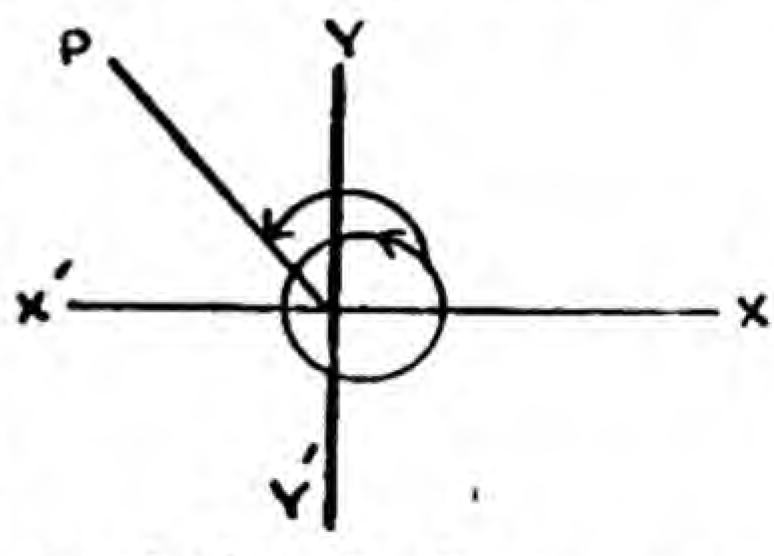
Quadrants. The perpendicular lines XOX′ and YOY′ divide the plane into four parts XOY, YOX′, X′OY′, and Y′OX, called the first, second, third and fourth quadrants respectively.

In the first quadrant the angle varies from $0°$ to $90°$
„ „ 2nd „ „ „ „ „ $90°$ to $180°$
„ „ 3rd „ „ „ „ „ $180°$ to $270°$
In the 4th quadrant the angle varies from $270°$ to $360°$

The angles XOP_1 and XOP_3 in the figure above are said to lie in the first and third quadrants respectively.

Ex. 1. In which quadrant does the revolviug line lie when it has turned tbrough (*i*) $500°$ (*ii*) $-1090°$ (*iii*) $780°$.

(*i*) Since $500° = 360° + 140°$ therefore the revolving line starting from OX will make one complete revolution and revolve further through an angle of $140°$. It will thus lie finally in the second quadrant.



(*ii*) $-1090 = -(3 \times 360° + 10°)$ therefore the revolving line starting from OX will make three complete revolution in the negative (*i. e.* the clockwise) direction and further move through an angle of $10°$ in the same direction. Thus the revolving line will finally lie in the fourth quadrant.

(*iii*) $780° = 720° + 60°$, therefore the revolving line starting from OX will make two complete revolutions and revolve further through an angle of $60°$ and lie finally in the 1st quadrant.

Ex. 2. By drawing figures, show in which quadrants do the following angles lie (*i*) $790°$ (*ii*) $-140°$ (*iii*) $-380°$.

[**Ans.** (*i*) 1st (*ii*) 3rd (*iii*) 4th]

Ex. 3. In which quadrant does the revolving line lie when it has turned through (*i*) $865°$ (*ii*) $270°$ (*iii*) $-840°$

[Ans. (*i*) 2nd (*ii*) coincides with OY′ (*iii*) 3rd.]

4. Different units for measuring angles. There are three systems for measuring angles in Trigonometry.

(1) The Sexagesimal or the English system.

(2) The Centesimal or the French system.

(3) The circular measure system.

In the Sexagesimal system a right angle is divided and sub-divided as follows :—

1 Right angle $=90$ degrees (written as $90°$)

1 degree $(1°)=60$ minutes (written as $60'$)

1 minute $(1')=60$ seconds (written as $60''$)

In the Centesimal system the sub-divisions are :—

1 right angle $=100$ grades (written as 100^g)

1 grade $=100$ minuter (written as $100^{\prime}$)

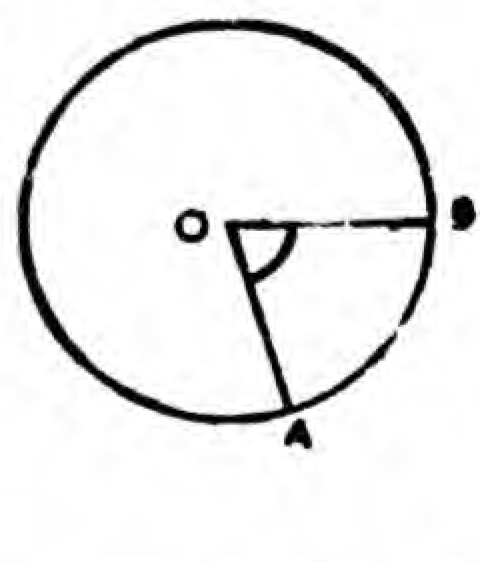
1 minute $=100$ seconds (wrltten as $100^{\prime\prime}$)

In the circular measure system the unit adopted is a **radian** **Radian is defined as an angle subtended at the centre of a circle by an arc equal in length to the radius of the circle.**

(*i. e.* $\angle$AOB in the figure)

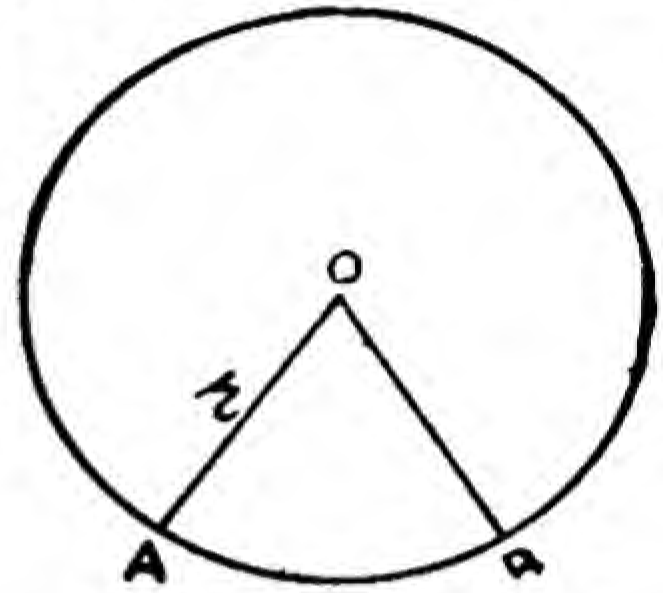
where arc AB $=$ radius OA

This system is used in higer branches of Mathematics.



The number of radians which an angle contains is the *circular measure of an angle*.

5. To prove that radian is a constant angle. (K.U.)



Draw any circle with centre O and radius r. Cut off an arc AB equal in length to the radius. Then $\angle AOB$ = a radian.

Since arcs of a circle are proportional to the angles they subtend at the centre.

$$\therefore \frac{\angle AOB}{4 \text{ rt } \angle s} = \frac{\text{arc AB}}{\text{circumference}}$$

$$i.e. \ \frac{1 \text{ radian}}{4 \text{ rt. } \angle s} = \frac{r}{2\pi r}$$

$\therefore$ one radian $= \frac{2}{\pi}$ rt. $\angle$s, which is a constant quantity.

Hence the radian is a constant angle.

Cor. $\because$ 1 radian $= \frac{2}{\pi}$ rt. $\angle$s

$\therefore$ **π radians = 2rt. $\angle$s = 180° = 200^g.**

This relation is useful in converting radians into degrees or grades and vice versa.

Ex $60° = 60 \times \frac{\pi}{180}$ *i.e.* $\frac{\pi}{3}$ radians.

Note. θ radians is denoted by θ^c. Thus π^c means π radians but it is usual to omit c or radians with π when speaking of an *angle* π. In that case the word *radian* is to be supplid mentally.

Ex. 1. To express the value of a radian in sexagesimal measure. (P. U.)

π radians = 180°

$$\therefore 1 \text{ radian} = \frac{180°}{\pi} = 180 \times \frac{113}{355} \text{ degrees}$$

$$= \frac{4068°}{7} = 57° \ 17' \ 44{\cdot}8''$$

$= 57° \ 17' \ 45''$ or $206265''$ approximately.

Ex. 2. If the angles of a triangle are in A. P., the least angle being 40°. Find all the angles in radians.

Let the angles in A P. be $(a-d)^\circ$, a°, $(a+d)^\circ$

$$\therefore (a-d)+a+(a+d)=180^\circ.$$

$$\text{or } 3a=180^\circ$$

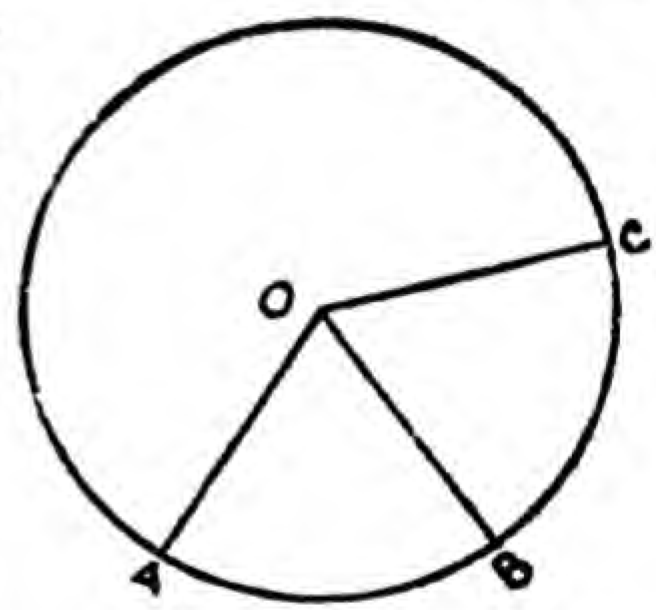
$$\text{or } a=60^\circ$$

Thus two of the angles are 40° and 60° and therefore the third is 80°.

The angles in radians are $\frac{40\pi}{180}$, $\frac{60\pi}{180}$ and $\frac{80\pi}{180}$

i. e. $\frac{2\pi}{9}$, $\frac{\pi}{3}$ and $\frac{4\pi}{9}$

6. To show that the circular measure of an angle subtended by an arc of a circle at the centre is equal to the length of the arc divided by the radius. (K U. 1958)



Let an arc AC of length l subtend an angle AOC$=\theta^c$ at the centre of a circle of radius r. Let arc AB be equal to the radius in length then $\angle$AOB$=$one radian.

Since the arcs of a circle are proportional to the angles they subtend at the centre of the circle,

$$\therefore \frac{\angle AOC}{\angle AOB}=\frac{\text{arc AC}}{\text{arc AB}}$$

or $\frac{\angle AOC}{1 \text{ radian}}=\frac{l}{r}$ *i. e.* $\angle AOC=\frac{l}{r}$ radians

$$\therefore \theta^c=\frac{l}{r}$$

Hence the circular measure of an angle

$$=\frac{\text{length of the arc}}{\text{radius of the circle}}$$

Ex. 1. Find the angle subtended at the centre of a circle of diameter 6 ft. by an arc 8 inches in length. Express the angle in degrees.

Here $r=3$ ft. and $l=$length of arc$=8''=\frac{2}{3}$ ft.

$$\therefore\quad \theta^c=\frac{l}{r}=\frac{\frac{2}{3}}{3}=\frac{2}{9}\text{ radians}$$

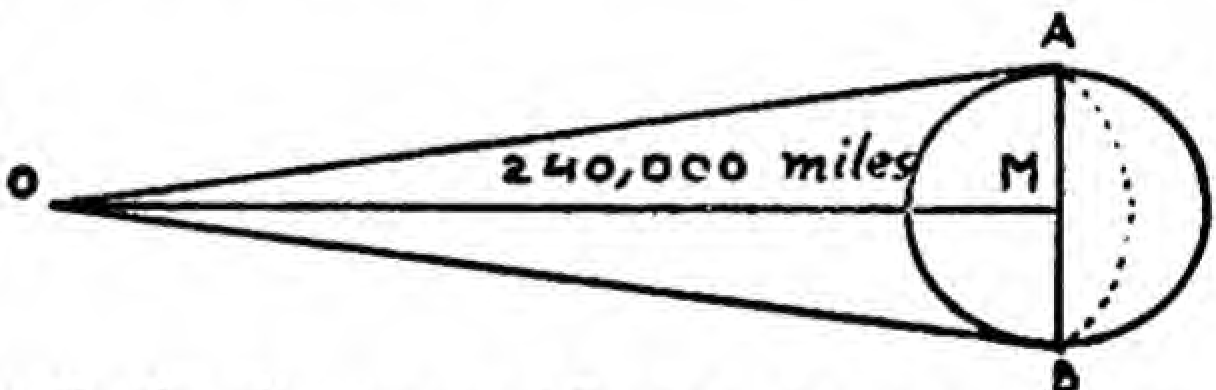
Now π radians$=180^\circ$

$$\therefore\quad \frac{2}{9}\text{ radians}=\frac{180}{\pi}\times\frac{2}{9}=180\times\frac{7}{22}\times\frac{2}{9}$$

$$=\frac{140}{11}\text{ i. e. } 12^\circ\ 43'\ 38.2''\text{ nearly}$$

Ex. 2. Find the diameter of the moon to the nearest mile, given that its disc subtends an angle of 30′ at the eye of an observer at a distance of 240,000 miles.

Let O be the observer and A B the diameter of the moon. With O as centre aud OA as radius draw an are AB. Since the



angle at O is very small, the diameter AB can be taken to be nearly equal to are AB and OM=OA.

Thus, here $l=$diameter AB.
$r=240{,}000$ miles.

$$\theta=30'=\frac{30}{60}\times\frac{\pi}{180}\text{ or }\frac{22}{7\times 360}\text{ radians}$$

$$\therefore\ \text{Diameter of the moon}=l=r\theta=240{,}000\times\frac{22}{7\times 360}$$

$$=\frac{44000}{21}=2095\text{ miles (nearly)}$$

Exercise 1.

1. Give the quadrants in which the revolving line would lie after turning through the angles :—

(*i*) 775° (*ii*) 1315^g (*iii*) $\frac{13\pi}{4}$ radians

2. Express the following angles in degrees :—

(*i*) $\frac{3\pi^c}{5}$ (*ii*) $2{\cdot}2$ radians (*iii*) $\frac{7\pi^g}{4}$

3. Express in radians the angles ;—

(*i*) 15° (*ii*) 60^g (*iii*) $71^\circ 15'$ (*iv*) $112^\circ 48'$

4. If the angles of a triangle be in A.P. and one of them be 95°, find all angles in radians. (P.U.)

5. If G, D, θ be the number of grades, degrees and radians in any angle, prove that :—

(*i*) $\frac{D}{9}=\frac{G}{10}=\frac{20\,\theta}{\pi}$ (J. & K. U. 1957)

(*ii*) $G-D=\frac{20\,\theta}{\pi}$ (P. U.)

6. Find the number of degree in the angle subtended at the centre of a circle of radius 10 ft. by an arc of length 20 ft. (J. & K. U.)

7. Express in radians and degrees the angle subtended at the centre of a circle by an arc of length 18 ft, when the radius of the circle is 30 ft.

8. Find the length of the arc which subtends an angle of 63° at the centre of a circle of radius 5 ft.

9. A pendulum 8 ft. long oscillates through an angle of 9°, what is the length of the path its extremity describes between the extreme positions ? (J. & K. U, 1958)

10. Assuming that the Earth's radius is 3960 miles and that it subtends an angle of 57′ at the centre of the moon, find the distance of the moon from the Earth's centre. (P. U.)

11. Meerut is 40 miles from Delhi. Find to the nearest second the angle subtended at the centre of the earth by the are joining these two towns, the earth being regarded as a sphere of 3960 miles radius. (D U. 1951)

12. Taking the radius of the earth to be 4000 miles, find the difference in the latitudes of two places, 200 miles apart, on the same meridian of longitude.

(Given, $\pi = \frac{22}{7}$) (J. & K. U. 1954)

CHAPTER II

Trigonometric Ratios.

1. Definition of Trigonometrical ratios.

Let a revolving line OP, starting from OX describe any angle XOP $(=\theta)$ of any magnitude in the anticlockwise direction so that OP is in any quadrant.

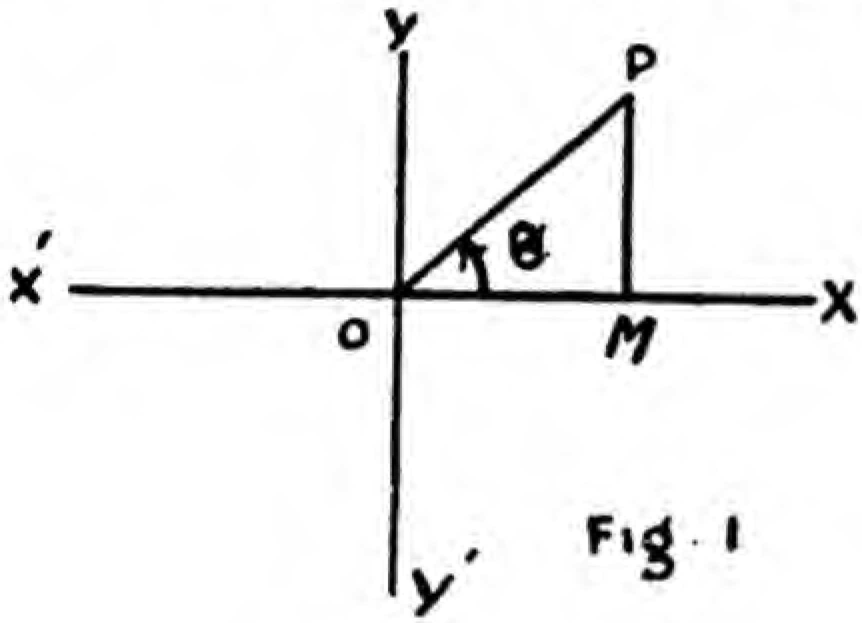


Fig. 1

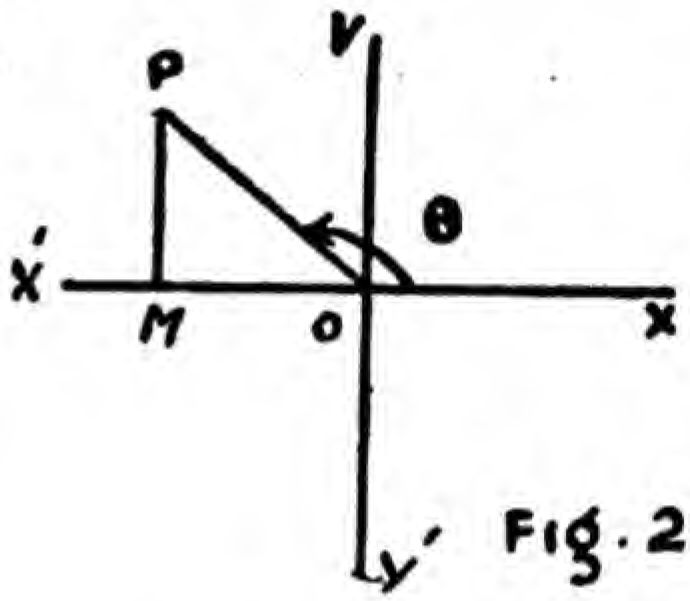


Fig. 2

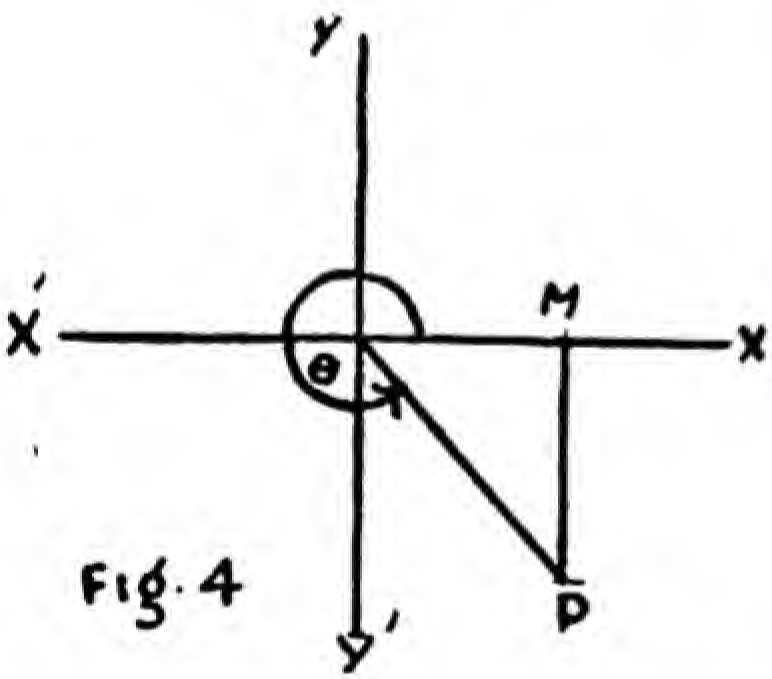


Fig. 4

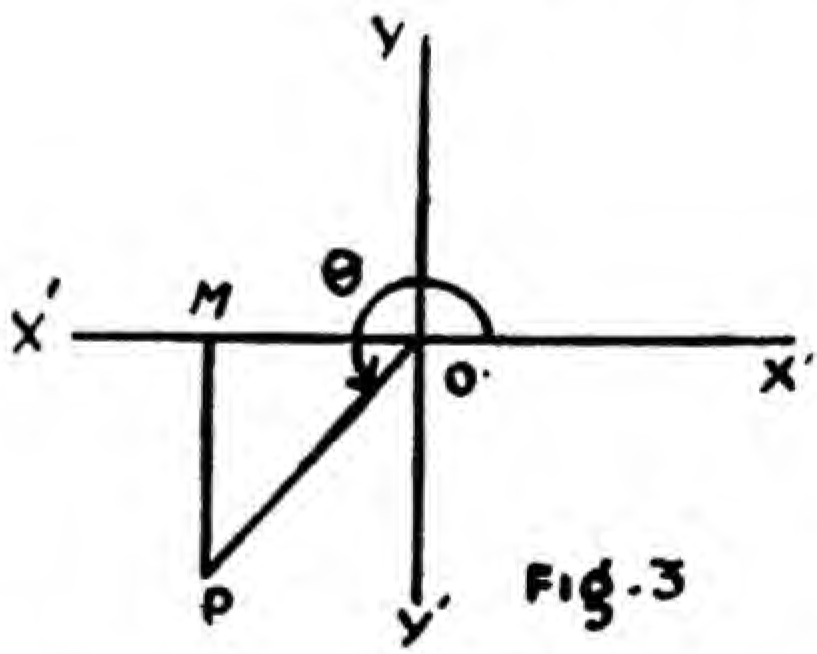


Fig. 3

Take any point P on OP and draw $PM \perp XOX'$.

Then, giving MP and OM their proper signs and taking OP always positive, the ratio

$\dfrac{MP}{OP}$ or $\dfrac{\textit{ordinate}}{\textit{Hpyotenuse}}$ is called the **sine** of angle θ and written as **sin** θ;

$\frac{OM}{OP}$ or $\frac{abscissa}{Hypotenuse}$ is called the **cosine** of angle θ and written as **cosθ**;

$\frac{MP}{OM}$ or $\frac{ordinate}{abscissa}$ is called the **tangent of angle** θ and written as **tan** θ;

$\frac{OM}{MP}$ or $\frac{abscissa}{ordinate}$ is called the **cotagent** of angle θ and written as **cot** θ;

$\frac{OP}{OM}$ or $\frac{Hypot.}{Abscissa}$ is called the **secant** of angle θ and written as **sec** θ;

$\frac{OP}{MP}$ or $\frac{Hypot.}{Ordinate}$ is called the **casecant** of angle θ and written as **cosec** θ;

The ratios defined above are called **trigonometric ratios or circular functions of** θ.

The triangle OMP is called the **triangle of reference for** the angle XOP.

Note 1. *In addition to the above six ratios there are two more. But these are rarely used.*

(*i*) $1-\cos\theta$ is known as **versed sine** of θ or briefly **vers** θ.

(*ii*) $1-\sin\theta$ is known as **coversed sine** of θ or briefly **covers** θ.

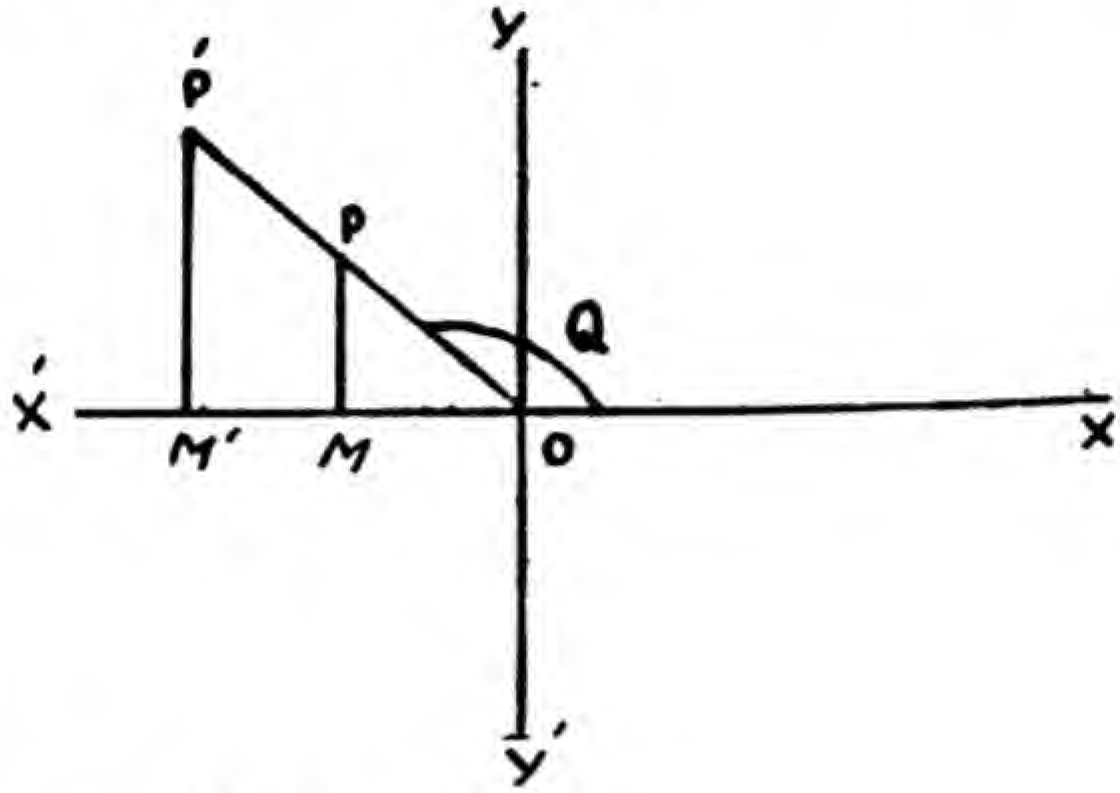
Note 2. *Sin θ does not mean sin × θ. It is a symbol only and stands for a certain ratio. Sine without an angle has no meaning.*

Note 3. *The above definitions are also true for angles described in the clockwise direction.*

2. The values of trigonometrical ratios are the same for the same angles.

Let the revolving line OP make an angle θ with the initial line OX Take any other point P′ on OP or OP

produced. Dyaw P′M′ and PM ⊥s to XOX′, then from similar triangles OMP and OM′P′,



$$\frac{M'P'}{OP'} = \frac{MP}{OP} = \sin\theta$$

Similarly $\dfrac{OM'}{OP'} = \dfrac{OM}{OP} = \cos\theta$

and so on for other ratios.

Hence the trigonometric ratios depend only upon the magnitude of the angle θ and not on the position of P on the line OP.

3. Relations between trigonometric ratios.

(*a*) The following simple relations follow from definitions given in art. 1.

(1) $\sin\theta \times \operatorname{cosec}\theta = \dfrac{MP}{OP} \times \dfrac{OP}{MP} = 1,$

$$\therefore \operatorname{Cosec}\theta = \frac{1}{\sin\theta} \text{ or } \sin\theta = \frac{1}{\operatorname{cosec}\theta}$$

(2) $\cos\theta \times \sec\theta = \dfrac{OM}{OP} \times \dfrac{OP}{OM} = 1,$

$$\therefore \sec\theta = \frac{1}{\cos\theta} \text{ or } \cos\theta = \frac{1}{\sec\theta}$$

(3) $\tan\theta \times \cot\theta = \dfrac{MP}{OM} \times \dfrac{OM}{MP} = 1,$

$$\therefore \cot\theta = \frac{1}{\tan\theta} \text{ or } \tan\theta = \frac{1}{\cot\theta}$$

(4) $\tan\theta = \dfrac{MP}{OM} = \dfrac{\dfrac{MP}{OP}}{\dfrac{OM}{OP}} = \dfrac{\sin\theta}{\cos\theta}$

(5) $\cot\theta = \dfrac{OM}{MP} = \dfrac{\dfrac{OM}{OP}}{\dfrac{MP}{OP}} = \dfrac{\cos\theta}{\sin\theta}$

(*b*) **Square relations. To prove that for all values of θ,**

(1) $\sin^2\theta + \cos^2\theta = 1$

(2) $1 + \tan^2\theta = \sec^2\theta$

(3) $1 + \cot^2\theta = \operatorname{cosec}^2\theta$

where $\sin^2\theta$, $\cos^2\theta$, $\tan^2\theta$.. ...stand for $(\sin\theta)^2$, $(\cos\theta)^2$, $(\tan\theta)^2$,.........

Let the revolving line OP starting from OX, trace out an $\angle XOP$ $(=\theta)$ of any magnitude *i. e* lie in any one of the four quadrants. (Draw the 4 figs. as in art. 1)

From any pt. P in OP draw $PM \perp XOX'$. Then in the rt. angled $\triangle OMP$, we have,

(*i*) $MP^2 + OM^2 = OP^2$......(A)

Dividing both sides of (A) by OP^2, we get

$$\left(\frac{MP}{OP}\right)^2 + \left(\frac{OM}{OP}\right)^2 = 1$$

or $\sin^2\theta + \cos^2\theta = 1$

(*ii*) Again dividing (A) by OM^2; we get

$$1 + \left(\frac{MP}{OM}\right)^2 = \left(\frac{OP}{OM}\right)^2$$

or $1 + \tan^2\theta = \sec^2\theta$.

(*iii*) Dividing (A) by MP^2 we get

$$1+\left(\frac{OM}{MP}\right)^2=\left(\frac{OP}{MP}\right)^2$$

Or, $\mathbf{1+\cot^2\theta=\operatorname{cosec}^2\theta}$.

Cor. 1. $\sin^2\theta=1-\cos^2\theta$ and $\cos^2\theta=1-\sin^2\theta$

Cor. 2. $\sec^2\theta-\tan^2\theta=1$ and $\sec^2\theta-1=\tan^2\theta$

Cor. 3. $\operatorname{cosec}^2\theta-\cot^2\theta=1$ and $\operatorname{cosec}^2\theta-1=\cot^2\theta$

The eight relations given above are very important.

Ex 1 Prove that $\cos^4 A-\sin^4 A=2\cos^2 A-1$

L H.S. $=(\cos^2A+\sin^2A)(\cos^2A-\sin^2A)=\cos^2A-\sin^2A$

$=\cos^2A-(1-\cos^2A)=2\cos^2 A-1=$ R.H.S.

Ex. 2. Prove that $\sin^2A+\tan^2A=\sec^2A-\cos^2A$.

To prove this is the same as to prove that

$\sin^2A+\cos^2A=\sec^2A-\tan^2A$ (transposing terms)

Now R H S. $=1+\tan^2A-\tan^2A=1=$ L.H.S.

Ex. 3. Prove that $\dfrac{\sin\theta}{1+\cos\theta}=\dfrac{1-\cos\theta}{\sin\theta}$

To prove this is the same as to prove that

$\sin^2\theta=1-\cos^2\theta$ (cross multiplying)

which is obvious from cor. 1 above

Ex. 4. Prove that $\sqrt{\dfrac{1+\cos A}{1-\cos A}}=\operatorname{cosec} A+\cot A$

$$\text{L H.S}=\sqrt{\frac{1+\cos A}{1-\cos A}}\times\sqrt{\frac{1+\cos A}{1+\cos A}}$$

$$=\frac{1+\cos A}{\sqrt{1-\cos^2A}}=\frac{1+\cos A}{\sin A}$$

$$=\frac{1}{\sin A}+\frac{\cos A}{\sin A}=\operatorname{cosec} A+\cot A.$$

Note. Sometimes it is easier to prove by expressing T-ratios in terms of sine and cosine as shown below.

Ex. 5. Prove that :— $\frac{1}{\sec A+\tan A}=\frac{1-\sin A}{\cos A}$

$$\text{L.H.S.}=\frac{1}{\frac{1}{\cos A}+\frac{\sin A}{\cos A}}=\frac{\cos A}{1+\sin A}$$

$$=\frac{\cos A}{1+\sin A}\times\frac{1-\sin A}{1-\sin A}$$

$$=\frac{\cos A\,(1-\sin A)}{\cos^2 A}=\frac{1-\sin A}{\cos A}$$

Ex. 6. Prove that $(1+\cot A-\operatorname{cosec} A)$
$(1+\tan A+\sec A)=2$

$$\text{L H.S}=\left(1+\frac{\cos A}{\sin A}-\frac{1}{\sin A}\right)\left(1+\frac{\sin A}{\cos A}+\frac{1}{\cos A}\right)$$

$$=\frac{\sin A+\cos A-1}{\sin A}\times\frac{\cos A+\sin A+1}{\cos A}$$

$$=\frac{(\sin A+\cos A)^2-1}{\sin A\cos A}$$

$$=\frac{\sin^2 A+\cos^2 A+2\sin A\cos A-1}{\sin A\cos A}$$

$$=\frac{1+2\sin A\cos A-1}{\sin A\cos A}=2=\text{R H S.}$$

Exercise 2.

Prove the following :—

1. $(\sin\theta+\cos\theta)^2=1+2\sin 0\cos\theta.$
2. $(1+\sin A)(1-\sin A)=\cos^2 A.$
3. $\operatorname{cosec}^3 A-1=\cos^2 A\operatorname{cosec}^2 A.$
4. $\sin^4 A+\cos^4 A=1-2\sin^2 A\cos^2 A.$
5. $\tan\theta\sin 0+\cos\theta=\sec\theta.$
6. $\tan\theta+\cot 0=\operatorname{cosec} 0 \text{ see } 0.$

7. $\dfrac{1}{\sec A+\tan A}-\dfrac{1}{\cos A}=\dfrac{1}{\cos A}-\dfrac{1}{\sec A-\tan A}.$

8. $\dfrac{1}{\sec A+\tan A}=\sec A-\tan A.$

9. $\dfrac{\sin^2 A}{1-\cos A}=1+\cos A$

10. $\sqrt{\dfrac{1-\sin A}{1+\sin A}}=\sec A-\tan A$ (J. & K. U. 1951)

11. $\dfrac{\sin A}{1+\cos A}+\dfrac{1+\cos A}{\sin A}=2 \operatorname{cosec} A.$

12. $\dfrac{\cos A}{1-\tan A}+\dfrac{\sin A}{1-\cot A}=\sin A+\cos A.$

13. $\dfrac{1}{1+\sin A}+\dfrac{1}{1-\sin A}=2 \sec^2 A.$ (D.U.)

14. $(\tan \theta+\cot \theta)^2=\sec^2 \theta+\operatorname{cosec}^2 \theta.$

15. $\sin^2 A \cos^2 B+\cos^2 B \sin^2 B+\sin^2 A \sin^2 B$
$+\cos^2 A \cos^2 B=1.$

16. $(\operatorname{cosec} A-\sin A)(\sec A-\cos A)(\tan A+\cot A)=1.$

17. $\dfrac{\tan A+\sec A-1}{\tan A-\sec A+1}=\dfrac{1+\sin A}{\cos A}$ (J. & K U. 1959)

18. $\dfrac{\tan A-\sec A+1}{\tan A+\sec A-1}=\sqrt{\dfrac{1-\sin A}{1+\sin A}}$ (J. & K.U. 1950)

19. $\sin^2 A\,(2+\tan^2 A)=\sec^2 A-\cos^2 A.$

20. $(1-\tan \theta)^2+(1-\cot \theta)^2=(\sec \theta-\operatorname{cosec} \theta)^2.$

21. $(\tan \theta+\sec \theta)^2=\dfrac{\operatorname{cosec} \theta+1}{\operatorname{cosec} \theta-1}$ (J. & K.U. 1958)

22. $\dfrac{\tan^2 \theta}{1+\tan^2 \theta}+\dfrac{\cot^2 \theta}{1+\cot^2 \theta}=\dfrac{1-2 \sin^2\theta \cos^2 \theta}{\sin \theta \cos\theta}$

23. $(\sin x+\sec x)^2+(\operatorname{cosec} x+\cos x)^2=(1+\sec x \operatorname{cosec} x)^2$
(J. & K.U. 1957)

24. $\dfrac{1-\sin A}{1-\sec A}-\dfrac{1+\sin A}{1+\sec A}=2 \cot A\,(\cos A-\operatorname{cosec} A)$

(ii) $\dfrac{1+\cos A}{1-\cos A}=(\text{cosec } A+\cot A)^2$ (J. & K.U. 1954)

25. Eliminate θ from the following equations :—

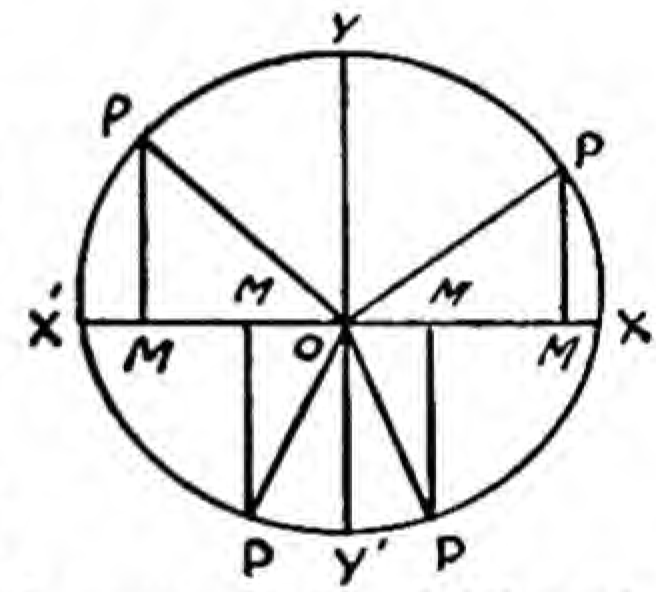
$x=a\cos\theta+b\sin\theta$

$y=a\sin\theta-b\cos\theta.$ (P.U. 1950)

[*Hint*. Solve the equations simultaneously for $\sin\theta$ and $\cos\theta$ and use the relation $\sin^2\theta+\cos^2\theta=1$.]

26. Eliminate θ from the equations $x=a\tan\theta$, $y=b\sec\theta$
[*Hint*. Find values of $\tan\theta$ and $\sec\theta$ and use $1+\tan^2\theta=\sec^2\theta$]

4. Signs of Trigonometrical ratios.



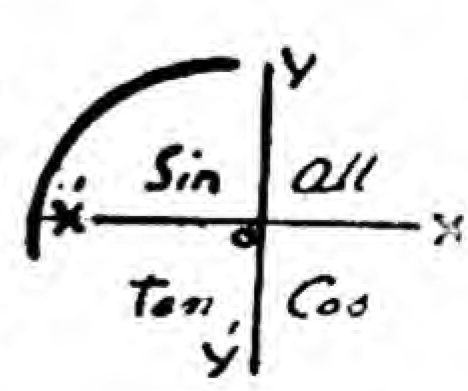
Quadrant I. In this quadrant the three lengths OM, MP and OP are positive. Hence the ratios containing any two lengths are positive. *Thus in the first quadrant all the six-T-ratios are positive.*

Quadrant II. In this quadrant only OM is negative, while OP and MP are both positive. Hence the T-ratios containing MP and OP are positive *i e. sin* θ and *cosec* θ *are positives*. All other will be negative.

Quadrant III. In this quadrant OM and MP are negative while OP is positive. Hence the T-ratios containing OM and OP are positive *i. e. tan* θ *and cot* θ *alone are positive.* All others will be negative.

Quadrant IV. In this quadrant MP is negative, while OM and OP are positive. Hence the T-ratios containing OM and OP are positive *i. e. cos* θ and *sec* θ *alone are positive* All others will be negative.

The above results can be summarised into one word **all—sin—tan—cos** by writing its parts (as shown in the figure) in the four quadrants in order : these parts indicate which T-ratio (and also its reciprocal) is positive for the marked quadrant.



5. Limits to the values of Trigonometric ratios.

Since OP is either greater than MP or OM or at the most equal to them therefore sin θ and cos θ are always numerically $\leqslant 1$ and consequently their reciprocals cosec θ and sec θ are $\geqslant 1$ numerically.

But because no restriction can be put on the lengths OM and MP, therefore tan θ and cot θ can have any values whatsoever.

or thus : $\sin^2\theta$ and $\cos^2\theta$. being squares, are positive and since $\sin^2\theta+\cos^2\theta=1$ *i. e.* sum of two positive numbers is unity, hence each of them is less than unity or at the most equal to unity.

$\therefore$ sin θ or cos θ cannot be numerically >1.

Hence their reciprocals cosec θ or sec θ cannot be numerically <1.

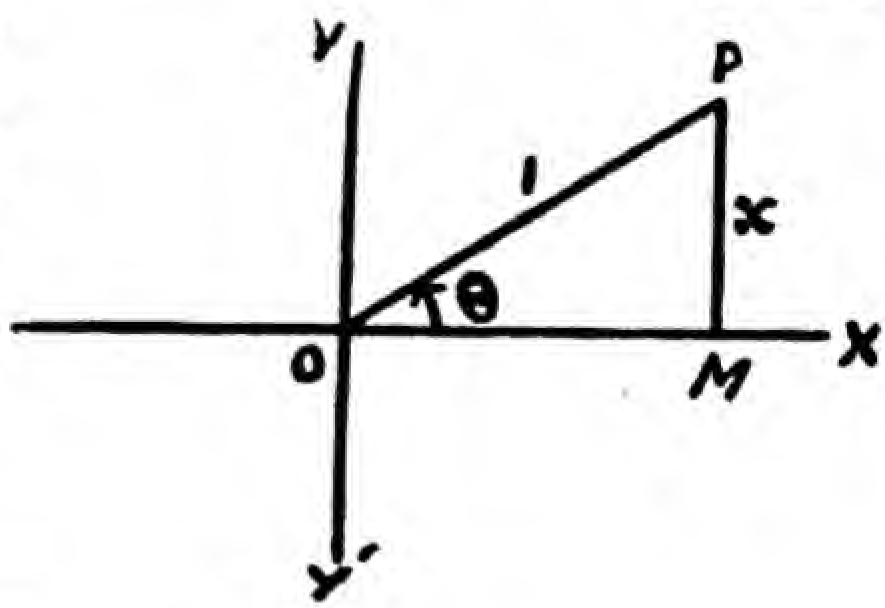
Again, because $\sec^2\theta=1+\tan^2\theta$ and sec θ is always $\geqslant 1$ hence tan θ can have any value.

Therefore its reciprocal cot θ can also have any value.

6. To express T-ratios of an angle in terms of any one of them.

Ex. 1. Express all the trignometrical ratios in terms of the sine of angle θ. (K.U.)

First Method. Let the revolving line OP, starting from OX trace out an $\angle XOP=\theta$. Cut off $OP=1$ and draw $PM \perp XOX'$.



If $\sin\theta=x$ (given)

then $\because$ $\sin\theta=\frac{MP}{OP}$

$\cdot\frac{MP}{OP}=x$ *i. e.* $MP=x(\because OP=1)$

$\because \triangle$ OMP is right angled, therefore $OM=\pm\sqrt{OP^2-MP^2}$
$=\pm\sqrt{1-x^2}$

Hence $\cos\theta=\dfrac{OM}{OP}=\pm\dfrac{\sqrt{1-x^2}}{1}=\pm\sqrt{1-\sin^2\theta}$

$$\tan\theta=\frac{MP}{OM}=\pm\frac{x}{\sqrt{1-x^2}}=\pm\frac{\sin\theta}{\sqrt{1-\sin^2\theta}}$$

$$\cot\theta=\frac{OM}{MP}=\pm\frac{\sqrt{1-x^2}}{x}=\pm\frac{\sqrt{1-\sin^2\theta}}{\sin\theta}$$

$$\sec\theta=\frac{OP}{OM}=\pm\frac{1}{\sqrt{1-x^2}}=\pm\frac{1}{\sqrt{1-\sin^2\theta}}$$

$$\operatorname{cosec}\theta\ \frac{OP}{MP}=\frac{1}{x}=\frac{1}{\sin\theta}$$

Note. In this method the ratios are found by constructing the rt. angled triangle corresponding to the given value of $\sin\theta$. The fig. is drawn for the case when θ is in the first quadrant but the method is general and can be used when the angle θ lies in any other quadrant.

Second Method. We can obtain the above results with the help of the formulae proved already.

$\because\ \sin^2\theta+\cos^2\theta=1$

$\therefore\ \cos\theta=\pm\sqrt{1-\sin^2\theta}$

$$\therefore\tan\theta=\frac{\sin\theta}{\cos\theta}=\pm\frac{\sin\theta}{\sqrt{1-\sin^2\theta}}$$

$$\cot\theta=\frac{\cos\theta}{\sin\theta}=\pm\frac{\sqrt{1-\sin^2\theta}}{\sin\theta}$$

$$\sec\theta=\frac{1}{\cos\theta}=\pm\frac{1}{\sqrt{1-\sin^2\theta}}$$

$$\operatorname{cosec}\theta=\frac{1}{\sin\theta}$$

Note. In the results the signs $\pm$ occur with rodicals. When nothing is said about the magnitude of the angle, the sign of the radicals is doubtful hence both signs must be taken. But when the magnitude

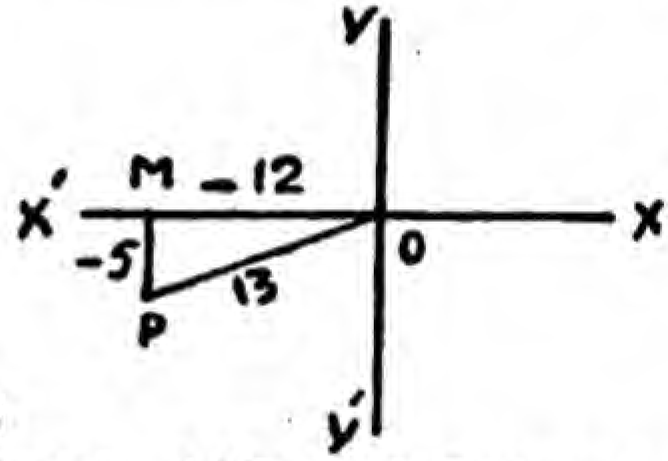
of θ is known we can find the proper signs of the trigonometric functions and attach it to the radical.

Rule. *First Step.* Put the given T-ratio$=x$ and put down its value in terms of the sides of the $\triangle$OMP.

2nd Step. Take the denominator$=1$. then the numerator$=x$ Now find third side of the rt. $\angle$d$\triangle$ OMP taking the signs$\pm$with the equare root.

3rd Step. Write down the values of the other T-ratios in terms of the sides of the $\triangle$ OMP and substitute the value of x.

Ex. 2. If $\sec\theta=-\frac{13}{12}$, where θ lies in the 3rd quadrant find the other circular functions θ.



Here $\sec\theta=-\frac{13}{12}$

But $\sec\theta=\frac{\text{OP}}{\text{OM}}$ in the fig.

Let OM$=-12$ and OP$=13$

so that $\text{MP}=-\sqrt{\text{OP}^2-\text{OM}^2}=-5$

As the angle lies in the 3rd quadrant, MP and OM are negative while OP is positive, hence all the functions except tangent and cotangent will be negative.

$$\therefore \sin\theta=\frac{\text{MP}}{\text{OP}}=\frac{-5}{13}\text{ and } \operatorname{cosec}\theta=\frac{-13}{5}$$

$$\cos\theta=\frac{\text{OM}}{\text{OP}}=\frac{-12}{13}\text{ and } \sec\theta=\frac{-13}{12}$$

$$\tan\theta=\frac{\text{MP}}{\text{OM}}=\frac{5}{12}\text{ and } \cot\theta=\frac{12}{5}$$

Ex. 3. Prove that the equation $\cos\theta=x+\frac{1}{x}$ is impossi. ble for real values of x.

$$\because \cos\theta=x+\frac{1}{x}$$

$$\because x^2-x\cos\theta+1=0$$

Now x is real if the discriminant is positive

i. e. if $\cos^2\theta - 4 > 0$

or $\cos^2\theta > 4$

which is impossible as $\cos\theta$ is never greater than unity. Hence the equation is impossible.

Ex. 4. If $\tan A = \frac{m}{n}$ prove that $\frac{m \sin A + n \cos A}{m \sin A - n \cos A}$

$$= \frac{m^2+n^2}{m^2-n^2}$$

Here $\frac{\sin A}{\cos A} = \frac{m}{n}$

$$\therefore \quad \frac{m}{n} \cdot \frac{\sin A}{\cos A} = \frac{m^2}{n^2}$$

Now by componendo and dividendo,

$$\frac{m \sin A + n \cos A}{m \sin A - n \cos A} = \frac{m^2+n^2}{m^2-n^2}.$$

Exercise 3.

1. What signs will the following have ?
   (*i*) $\sin 130^\circ$ (*ii*) $\sec 140^\circ$ (*iii*) $\tan 310^\circ$
   (*iv*) $\cos(-320^\circ)$ (*v*) $\cot \frac{2\pi}{3}$.
2. Express all the circular functions of θ in terms of $\cos\theta$.
3. Express all circular functions in terms of $\sec\theta$.
4. If $\cos A = \frac{3}{7}$ and A lies in the fourth quadrant, find cot A and cosec A.
5. If $\operatorname{cosec} A = -\frac{13}{12}$ and A lies in the third quadrant, find the value of $\sin A + \tan A$.
6. Examine whether the following are possible :
   (*i*) $\sin\theta = \frac{4}{5}$ (*ii*) $\cos\theta = \frac{9}{8}$ (*iii*) $\tan\theta = 80$
   (*iv*) $\sin\theta = \frac{a^2+b^2}{a^2-b^2}$ (*v*) $\sec\theta = \frac{a^2+b^2}{2ab}$.
7. Is the equation $3\sin^2 + 5\sin\theta - 2 = 0$ possible ?
8. Can an angle θ exist such that $9\sec^2\theta + 6\tan\theta = -1$?

9. Prove that $\sin\theta = x + \frac{1}{x}$ is not possible for real values of x.

10. If $\cot\theta = \frac{a}{b}$, show that $\frac{a\cos\theta - b\sin\theta}{a\cos\theta + b\sin\theta} = \frac{a^2 - b^2}{a^2 + b^2}$

11. In which quadrant will θ lie when

(*i*) $\sin A = \frac{1}{\sqrt{6}}$ and $\cos A = -\sqrt{\frac{5}{6}}$

(*ii*) $\cot\theta = -3$ and $\sec\theta = \frac{\sqrt{10}}{3}$

12. If $\tan\theta = \frac{1}{\sqrt{3}}$, find the quadrants in which θ can lie. Find the other Trigonometrical ratios also. (P. U. 1945)

13. If $6\cos^2\theta = 1$, find all trigonometric ratios. (P. U. 1947)

14. If $\cos A = 2\sin A$, find cosec A. (D. U. 1946)

15. If A be an angle in the second quadrant, and $\sin A = \frac{5}{13}$ find the value of.

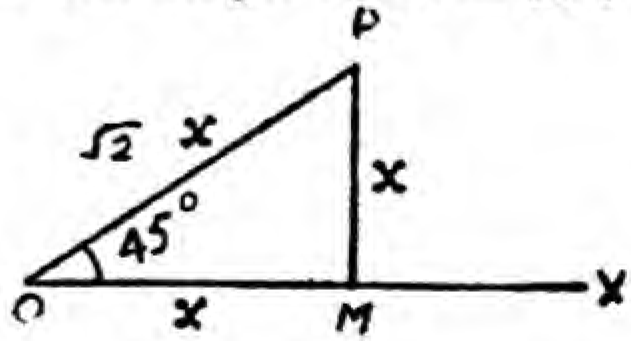
$$\frac{5\cot A - 4\sec A}{\cos A + \sin A}$$ (J. & K. U. 1954)

16. If A is in the 4th quadranr and $\cos A = \frac{5}{13}$, find the value of $\frac{13\sin A + 5\sec A}{\tan A + 6\operatorname{cosec} A}$

The Trigonometric ratios of some well known angles.

7. The Trigonometric ratios of 45° or $\frac{\pi}{4}$.

Let OP starting from OX trace out an angle $XOP = 45°$. From any point P in OP draw $PM \perp OX$. Then $\angle OPM = 45°$



$\therefore OM = MP = x$ (say)

$\because \triangle OMP$ is rt. $\angle$d.

$\therefore OP^2 = OM^2 + MP^2 = 2x^2$

or $OP=\sqrt{2}x$ (Taking+ve sign as the angle is in the 1st Quadrant).

$$\therefore \sin 45^\circ=\frac{MP}{OP}=\frac{x}{\sqrt{2}x}=\frac{1}{\sqrt{2}}$$

$$\cos 45^\circ=\frac{OM}{OP}=\frac{x}{\sqrt{2}x}=\frac{1}{\sqrt{2}}$$

$$\tan 45^\circ=\frac{MP}{OM}=\frac{x}{x}=1$$

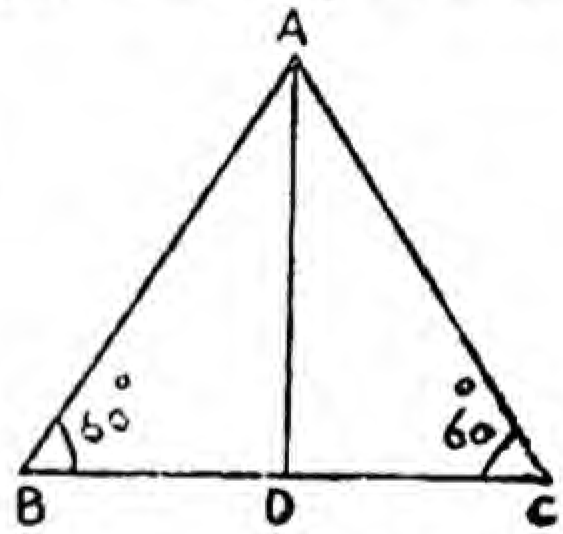
$$\sec 45^\circ=\frac{OP}{OM}=\frac{\sqrt{2}x}{x}=\sqrt{2}$$

$$\operatorname{cosec} 45^\circ=\frac{OP}{MP}=\frac{\sqrt{2}x}{x}=\sqrt{2}$$

$$\cot 45^\circ=\frac{OM}{MP}=\frac{x}{x}=1$$

8. The Trigonometrical ratios 30° or $\frac{\pi}{6}$.

Let us first show that in a rt. ∠d △, the side opposite to an angle of 30° is half of the hypotenuse. Take an equilateral △ ABC and draw AD⊥BC, then ∠BAD=∠CAD=30°



∵ △s ABD and ACD are congruent obviously.

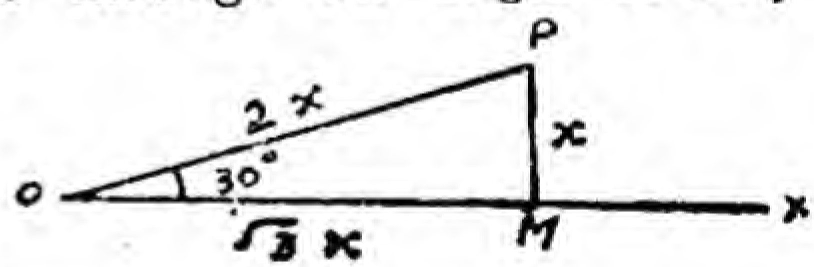
∴ $BD=DC=\frac{1}{2}AB$.

Now we shall find trigonometric ratios of 30°.

Let OP, starting from OX, revolve through an angle of 30°, so that ∠XOP=30°.

From any point P in OP draw PM⊥OX.

Let $MP=x$ then $OP=2x$



(Proved above)

and $OM=\sqrt{OP^2-MP^2}=\sqrt{4x^2-x^2}=\sqrt{3}x$

(Taking+ve sign as the angle is in the first quadrant)

$$\therefore \sin 30^\circ = \frac{MP}{OP} = \frac{x}{2z} = \frac{1}{2}$$

$$\cos 30^\circ = \frac{OM}{OP} = \frac{\sqrt{3}x}{2x} = \frac{\sqrt{3}}{2}$$

$$\tan 30^\circ = \frac{MP}{OM} = \frac{x}{\sqrt{3}} = \frac{1}{\sqrt{3}}$$

Similarly $\cot 30^\circ = \sqrt{3}$, $\sec 30^\circ = \frac{2}{\sqrt{3}}$ and $\operatorname{cosec} 30^\circ = 2$

9. Trigonometrical ratios of 60° or $\frac{\pi}{3}$.

Let OP, starticg from OX, revolve through an angle of 60° so that $\angle XOP = 60^\circ$.

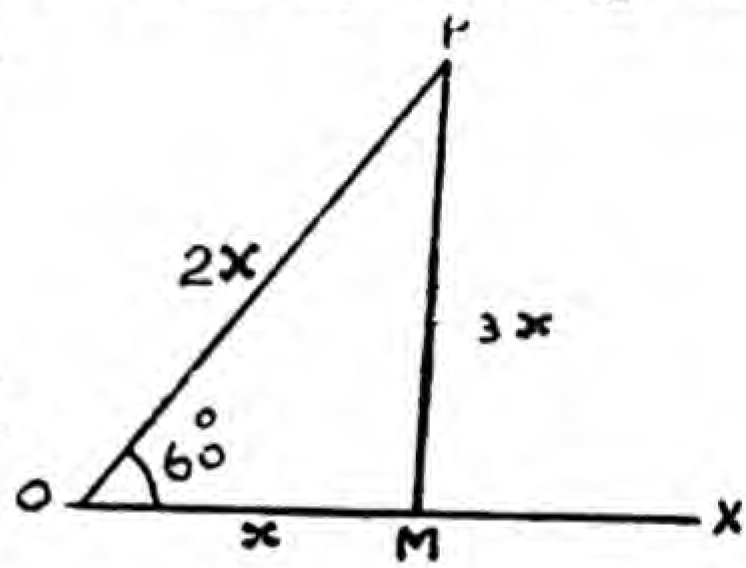
From any point] P in OP draw $PM \perp OX$.

Let $MO = x$ then $OP = 2x$

and $MP = \sqrt{OP^2 - OM^2}$

$= \sqrt{4x^2 - x^2} = \sqrt{3}x$

$$\therefore \sin 60^\circ = \frac{MP}{OP} = \frac{\sqrt{3}x}{2x} = \frac{\sqrt{3}}{2}$$



$$\cos 60^\circ = \frac{OM}{OP} = \frac{x}{2x} = \frac{1}{2}$$

$$\tan 90^\circ = \frac{MP}{OM} = \frac{\sqrt{3}x}{x} = \sqrt{3}$$

Similarly $\cot 60^\circ = \frac{1}{\sqrt{3}}$, $\sec 60^\circ = 2$, $\operatorname{cosec} 60^\circ = \frac{2}{\sqrt{3}}$

10. Before finding the trigonometrical ratios of 0° and 90° the student should clearly grasp the meanings of **'infinity'** and **'zero'**.

Def. Infinity is a number larger than any that can be named or conceived. The symbol for infinity is ∞.

Consider the fraction $\frac{a}{x}$, where a has a positive finite value.

Let x be given positive values which decrease steadily to 0, *e. g.* $\frac{1}{10}$, $\frac{1}{100}$, $\frac{1}{1000}$, $\frac{1}{10000}$,..................Then $\frac{a}{x}$ takes the values $10a$, $100a$, $1000a$, $10000a$,......which increase continually and without limit. We express this by saying that the limit of $\frac{a}{x}$, when x tends to zero (remaining positive always) is infinity and write $\underset{x \to 0}{\text{Lt.}} \frac{a}{x} = \infty$, or $\frac{a}{+0} = \infty$

Similarly, the limit of $\frac{a}{x}$, when x (remaining negative) tends to zero, is minus infinity *i. e.* $\frac{a}{-0} = -\infty$.

Note. Infinity is not an ordinary number and so an equation of the form $x = \infty$ is meaningless.

There are two conceptions of zero ; (*i*) absolute nothing and (*ii*) an indenfinitely small quantity. The second conception is important and is explained below :—

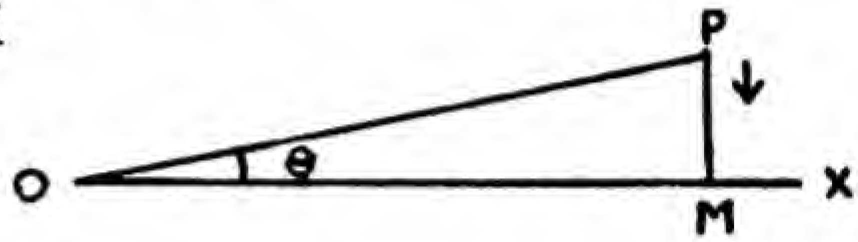
Def. Zero is a number which is smaller than any assignable fraction of unity. A quanity is finite when it is neither zero nor infinitely large. Consider the positive fraction $\frac{a}{x}$ where a is a positive fixed quantity and x takes only positive values. Give to x values 10, 100, 1000, 10000,........ then the fraction $\frac{a}{x}$ takes the values $\frac{a}{10}$, $\frac{a}{100}$, $\frac{a}{1000}$....... which decrease continually and without limit. So that as x becomes greater and greater and approaches $+\infty$, $\frac{a}{x}$ becomes smaller and smaller and approaches zero. This is written as

$$\underset{x \to \infty}{\text{Lt.}} \frac{a}{x} = +0, \text{ or } \frac{a}{\infty} = 0.$$

Similarly when x approaches $-\infty$, $\frac{a}{x}$ approaches -0 *i. e.*

Zero and this is written as $\underset{x \to -\infty}{\text{Lt.}} \frac{a}{x} = 0$ or $\frac{a}{-\infty} = 0$.

11. The Trigonometrical ratios of 0°.



Let OP starting from OX revolve through a small angle θ.

From any point P in OP draw PM ⊥ OX

When $\angle XOP = 0°$, OP coincides with OX, and P with M so that $OP = OM = x$ (say)

and $MP = 0$.

$$\text{Hence } \sin 0° = \frac{MP}{OP} = \frac{0}{x} = 0$$

$$\cos 0° = \frac{OM}{OP} = \frac{x}{x} = 1$$

$$\tan 0° = \frac{MP}{OM} \; \frac{0}{x} = 0$$

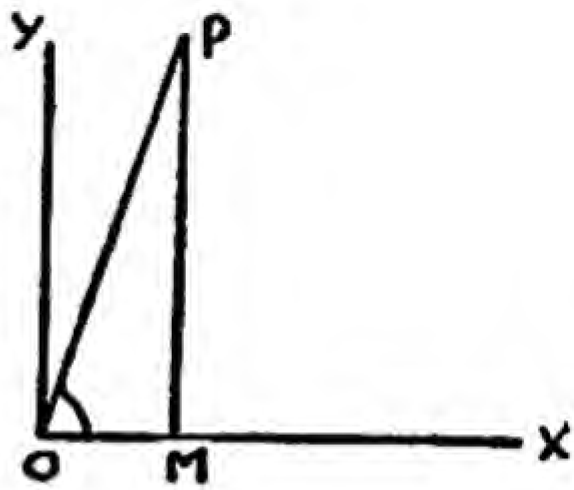
$$\cot 0° = \frac{OM}{MP} = \frac{x}{0} = \infty$$

$$\sec 0° = \frac{OP}{OM} = \frac{x}{x} = 1$$

$$\text{cosec } 0° = \frac{OP}{MP} = \frac{x}{0} = \infty$$

The values obtained are really the limiting values of the ratios when the angle tends to 0° through +ve values. The T-ratios when the angle tends to zero through negative values can be obtained similarly by drawing another figure in which OP will be below OX. In that case all the T-ratios are same except cot $(-0°)$ and cosec $(-0°)$ each of these being equal to $-\infty$.

12. The Trigonometrical ratios of 90°



Let OP revolve through an $\angle XOP$ which is a little less than 90°. From any point P in OP draw $PM \perp OX$.

When $\angle XOP = 90°$, OP coincides with OY and M with O.

$\therefore OP = OY = x$ (say)

$MP = OY = x$ and $OM = 0$.

$$\text{Hence } \sin 90° = \frac{MP}{OP} = \frac{x}{x} = 1$$

$$\cos 90° = \frac{OM}{OP} = \frac{0}{x} = 0$$

$$\tan 90° = \frac{MP}{OM} = \frac{x}{0} = \infty$$

$$\cot 9° = \frac{OM}{MP} = \frac{0}{x} = 0$$

$$\sec 90° = \frac{OP}{OM} = \frac{x}{0} = \infty$$

$$\operatorname{cosec} 90° = \frac{OP}{MP} = \frac{x}{x} = 1$$

The values obtained are really the limiting values of the T-ratios of an acute angle when the angle tends to 90° throgh acute values. The T-ratios when the angle tends to 90° through obtuse vaules can be obtained similarly by drawing another figure in which OP will be to the left of OY. In this case all the T-ratios are same except tan (+90°) and sec (+90°). each of these being equal to $-\infty$

The value of the T-ratios found above can be easily

remembered with the help of the following table :—

Angle	0°	30°	45°	60°	90°
Sine	$\sqrt{\frac{0}{4}}$	$\sqrt{\frac{1}{4}}$	$\sqrt{\frac{2}{4}}$	$\sqrt{\frac{3}{4}}$	$\sqrt{\frac{4}{4}}$
Cosine	$\sqrt{\frac{4}{4}}$	$\sqrt{\frac{3}{4}}$	$\sqrt{\frac{2}{4}}$	$\sqrt{\frac{1}{4}}$	$\sqrt{\frac{0}{4}}$
Tangent	$\sqrt{\frac{0}{4-0}}$	$\sqrt{\frac{1}{4-1}}$	$\sqrt{\frac{2}{4-2}}$	$\sqrt{\frac{3}{4-3}}$	$\sqrt{\frac{4}{4-4}}$

Ex. 1. Solve the equation $3 \tan \theta + \cot \theta = 5 \operatorname{cosec} \theta$.

Here we have to find the value of θ. The equation can be written as

$$3 \frac{\sin \theta}{\cos \theta} + \frac{\cos \theta}{\sin \theta} = 5 . \frac{1}{\sin \theta}$$

or $3 \sin^2\theta + \cos^2\theta = 5 \cos \theta$

or $3 (1 - \cos^2\theta) + \cos^2\theta = 5 \cos \theta$

or $2 \cos^2\theta + 5 \cos \theta - 3 = 0$, which is quadratsc in $\cos \theta$.

$$\therefore \cos \theta = \frac{-5 \pm \sqrt{25+24}}{4} = \frac{-5 \pm 7}{4} = \tfrac{1}{2} \text{ or } -3$$

Now $\cos \theta = \frac{1}{2}$ gives $\theta = 60°$,

but $\cos \theta = -3$ is impossible (as cosine of an angle is never > 1 numerically)

Exercise 4.

Prove the following :—

1. $\text{Sin } 60° \cos 30° - \cos 60° \sin 30° = \sin 30°$
2. $\text{Sin } 45° \cos 30° + \cos 45° \sin 30° = \cos 45° \cos 30° + \sin 45° \sin 30°$
3. $\sec^2 60° + \cot^2 45° + \cos 60° = \frac{11}{2}$

4. $\tan 60^\circ = \dfrac{2 \tan 30^\circ}{1-\tan^2 30^\circ}$

5. $4 \tan 45^\circ - \operatorname{cosec}^2 30^\circ + \sin^2 60^\circ = \dfrac{3\sqrt{3}}{8}$

If $A=30^\circ$, $B=45^\circ$, $C=60^\circ$, find the values of following :—

6. $\sin^2 A + \sin^2 C$

7. $\cos B \sin B - \sin^2 A$

8. $\dfrac{\sec A}{\operatorname{cosec} B} - \dfrac{\sec B}{\cot A}$

9. $\dfrac{2 \tan A}{1-\tan^2 A} - \tan C$

10. $\cos A \cos B - \sin A \sin B$

Solve the following equations :—

11. $2 \sin^2\theta + 5 \sin \theta - 3 = 0$

12. $2 \sin^2 \theta = 3 \cos \theta$

13. $\cot \theta = 2 \cos \theta$

14. $\operatorname{cosec}^2 \theta + \sqrt{3} \cot \theta - 7 = 0$

15. If $\sin (A-B) = \frac{1}{2}$ and $\cos (A+B) = \frac{1}{2}$, find the acute values of A and B.

16. If $\tan (A+B) = \sqrt{3}$ and $\tan (A-B) = 1$, find acute values of A and B.

17. Prove that (*i*) $\sin \dfrac{\pi}{4} \cos \dfrac{\pi}{6} \tan \dfrac{\pi}{3} \sec \dfrac{\pi}{3} = \dfrac{3}{\sqrt{2}}$

(*ii*) $\cos^2 A + \cos^2 B + \cos^2 C + 2 \cos A \cos B \cos C = 1$, when $A=30^\circ$, $B=60^\circ$, and $C=90^\circ$.

19. Find the values of

(*i*) $\operatorname{cosec} \dfrac{\pi}{4} \operatorname{cosec} \dfrac{\pi}{3} \left(\operatorname{cosec} \dfrac{\pi}{6} - \operatorname{cosec} \dfrac{\pi}{2} \right)$

(*ii*) $\dfrac{(\sin 60^\circ + \cos 30^\circ)(\sin 30^\circ + \tan 45^\circ)}{(\tan 30^\circ + \tan 60^\circ)(\sec 60^\circ - \operatorname{cosec} 90^\circ)}$

———

CHAPTER III

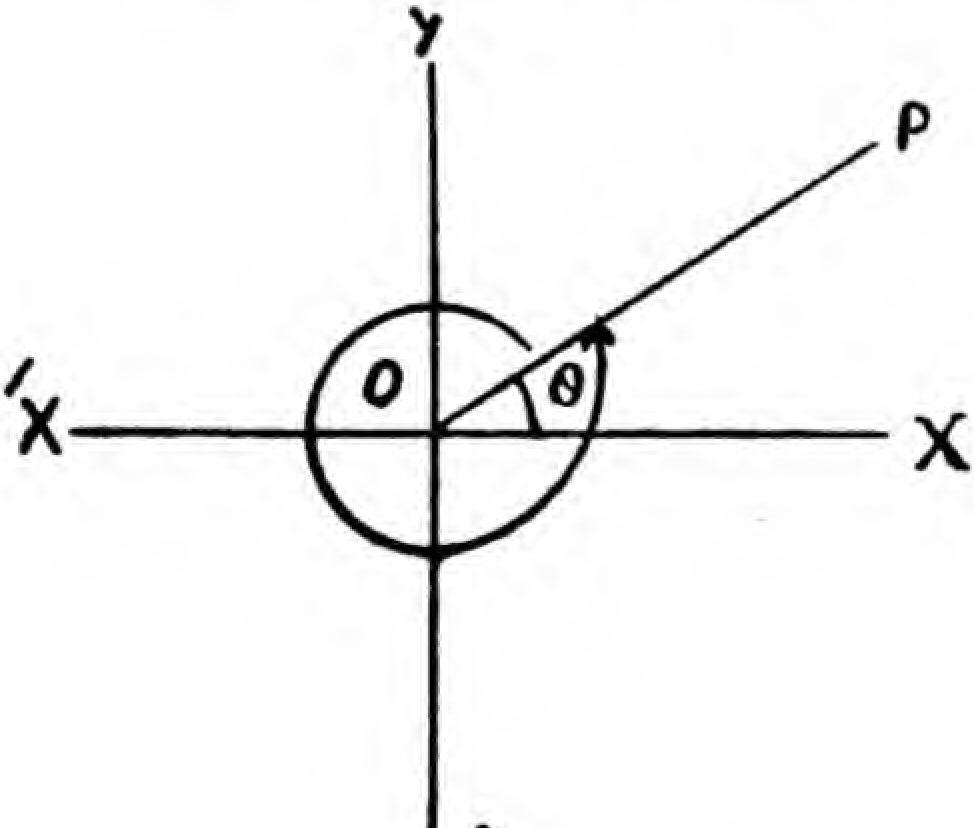
Trigonometric Ratios of Allied angles.

Def. The angles $-\theta$, $90°+\theta$, $90°-\theta$, $180°+\theta$, $180°-\theta$, $360°+\theta$. $360°-\theta$,.........are called angles allied to θ, when θ is given in degrees.

1. To show that the T-ratios of angles $\theta \pm 360$ are the same as those of θ.

From the definition of Trigonometric ratios it is clear that the final position of the revolving line OP after revolving through θ or $\theta+360°$ or $\theta-360°$ is the same. Since the values of T-ratios of any angle depend only on the final position of the revolving line, therefore the T-ratios of any angle θ and those of $\theta \pm 360°$ are equal.

In particular, $\sin(\theta+360°)=\sin\theta=\sin(\theta-360°)$
and $\cos(\theta+360°)=\cos(\theta-360°)$



Note. If we add to θ or subtract from θ any multiple of 360°, even then the final position of the revolving line remains unchanged. Therefore the T-ratios of all such angles are equal,

and $\sin(\theta+2n\,\pi)=\sin\theta=\sin\;(\theta-2n\,\pi)$ where n is any integer $2\,\pi$ means $2\,\pi$ radians.

Similarlo for other T-ratios.

Cor. (*i*) $\sin 360°=\sin(360°+0°)=\sin 0°=0$

(*ii*) $\cos 360°=\cos(360°+0°)=\cos 0°=1.$

2. To find the T-ratios of $-\theta$ in terms of those for θ, for all values of θ.

Let the revolving line OP, starting from OX, revolve through an $\angle XOP=\theta$, lying in any of the four quadrants.

Let another revolving line OP′ (=OP), starting from OX revolve in the opposite direction so that $\angle XOP'=-\theta$.

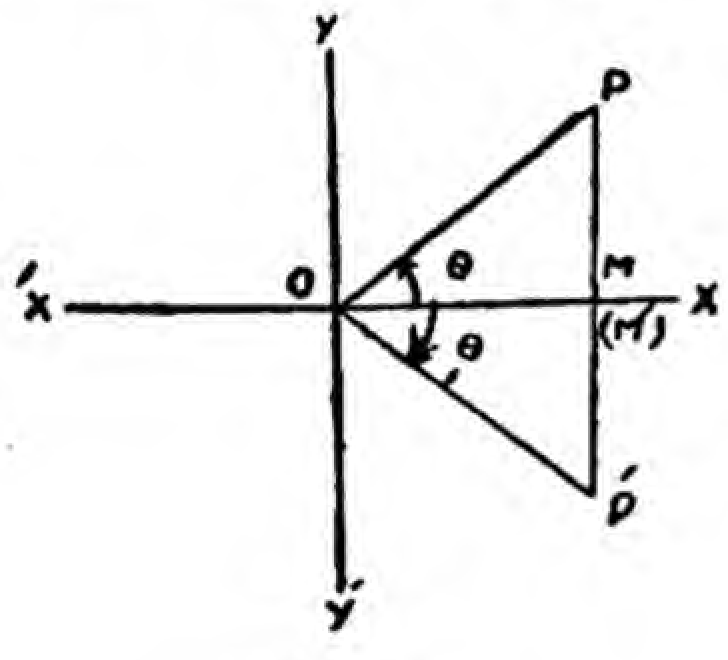


Fig. 1.

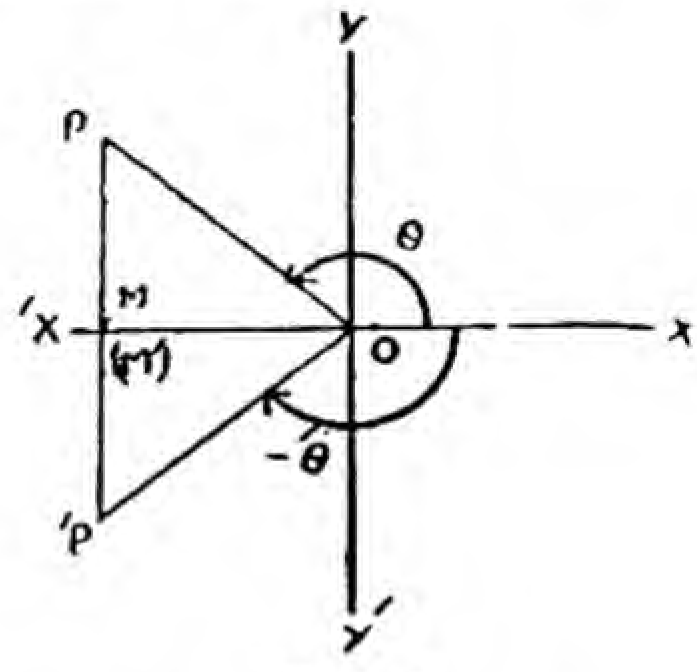


Fig. 2.

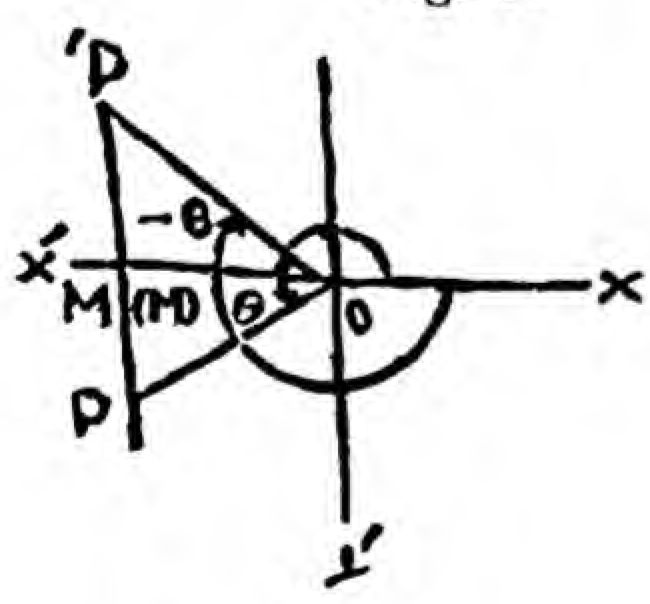


Fig. 3.

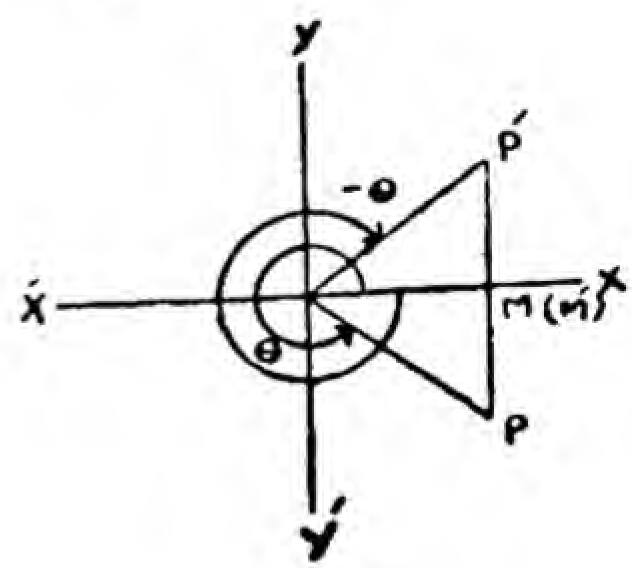


Fig. 4.

Draw PM and P′M′ ⊥s to XOY′.

Then in △s OMP add OM′P′,

$$OP=OP', \qquad \angle MOP=\angle M'OP'$$
$$\angle OMP=\angle OM'P'=1 \text{ rt. } \angle.$$

$\therefore$ $\triangle$s are congruent. Therefore having regard to the sings of the lines, we have $OM'=OM$

$$M'P'=-MP$$

$$\therefore \sin(-\theta)=\frac{M'P'}{OP'}=\frac{-MP}{OP}=-\sin\theta$$

$$\cos(-\theta)=\frac{OM'}{OP'}=\frac{OM}{OP}=\cos\theta$$

$$\tan(-\theta)=\frac{M'P'}{OM}=\frac{-MP}{OM}=-\tan\theta$$

$$\cot(-\theta)=\frac{OM'}{M'P'}=\frac{OM}{-MP}=-\cot\theta$$

$$\sec(-\theta)=\frac{OP'}{OM'}=\frac{OP}{OM}=\sec\theta$$

$$\operatorname{cosec}(-\theta)=\frac{OP'}{M'P'}-\frac{OP}{-MP}=\frac{-OP}{MP}=-\operatorname{cosec}\theta.$$

2. To find the T-ratios of $180°-\theta$ in terms of those of θ for all values of θ.

Let the revolving line OP starting from OX, revolve through an $\angle XOP=\theta$, lying in any quarant.

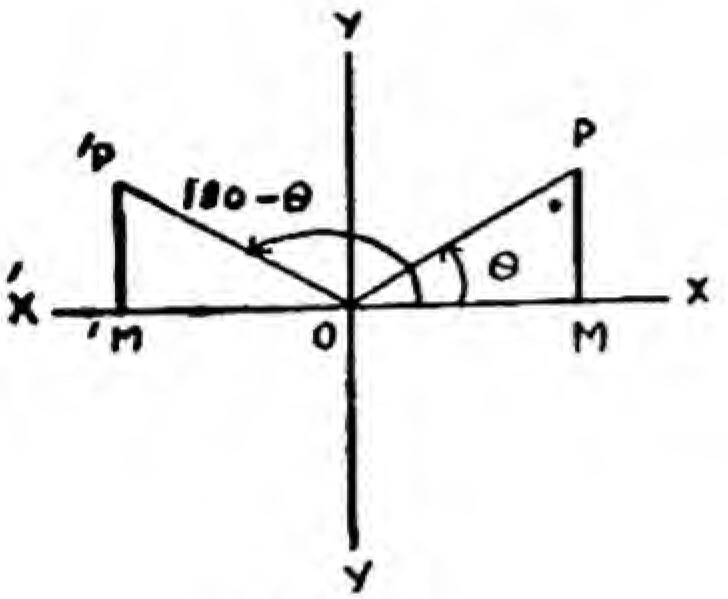


Fig 1

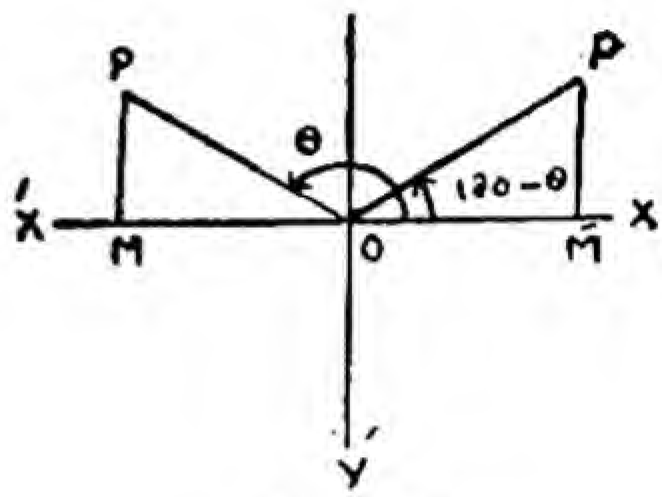


Fig. 2

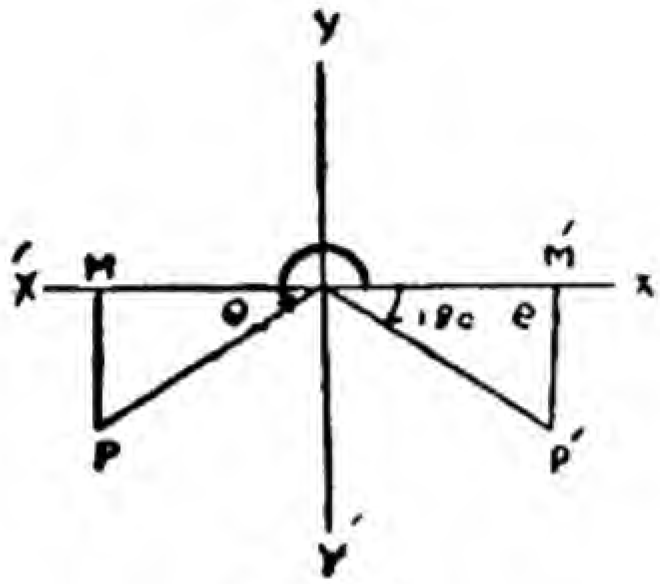


Fig. 3

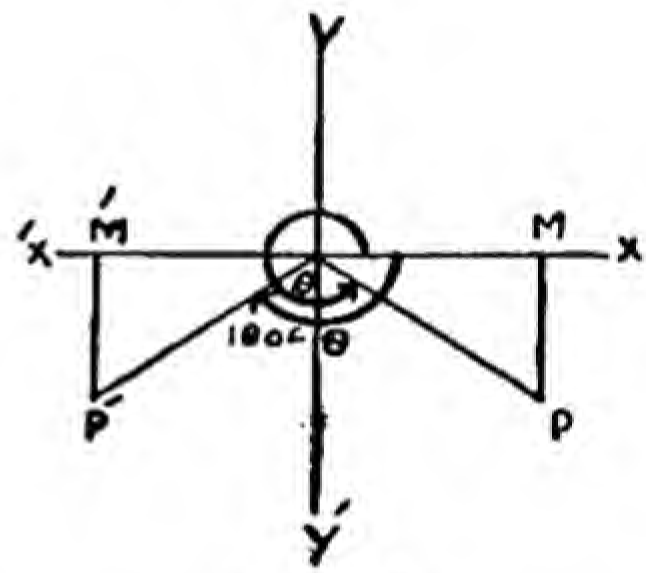


Fig. 4

Let another line OP′ (=OP), starting from OX, first revolve through 180° and then revolve back through θ, so that $\angle XOP' = 180° - \theta$.

Draw PM and P′M′ $\perp$s. to XOX′.

Then in $\triangle$s OMP and OM′P′,

$$OP = OP', \qquad \angle MOP = \angle M'OP',$$

$$\angle OMP = \angle OM'P' = 1 \text{ rt. } \angle.$$

$\therefore$ $\triangle$s are congruent. Having regard to the signs of the lines, we have

$$OM' = -OM, \; M'P' = MP.$$

$$\therefore \sin(180° - \theta) = \frac{M'P'}{OP'} = \frac{MP}{OP} = \sin\theta$$

$$\cos\,(180^\circ-\theta)=\frac{OM'}{OP'}=\frac{-OM}{OP}=-\cos\theta$$

$$\tan\,(180^\circ-\theta)=\frac{M'P'}{OM'}=\frac{MP}{-OM}=-\tan\theta$$

$$\cot\,(180^\circ-\theta)=\frac{OM'}{M'P'}=\frac{-OM}{MP}=-\cot\theta$$

$$\sec\,(180^\circ-\theta)=\frac{OP'}{OM'}=\frac{OP}{-OM}=-\sec\theta$$

$$\operatorname{cosec}\,(180^\circ-\theta)=\frac{OP'}{M'P'}=\frac{OP}{MP}=\operatorname{cosec}\theta$$

4. To find T-ratios of $(180^\circ+\theta)$ in terms of those of θ, for all values of θ.

Let the revolving line OP, starting from OX, revolve through $\angle XOP=\theta$.

Let another revolving line OP' (=OP). starting from OX, revolve through 180° and then revolve further through θ so that $\angle XOP'=180^\circ+\theta$.

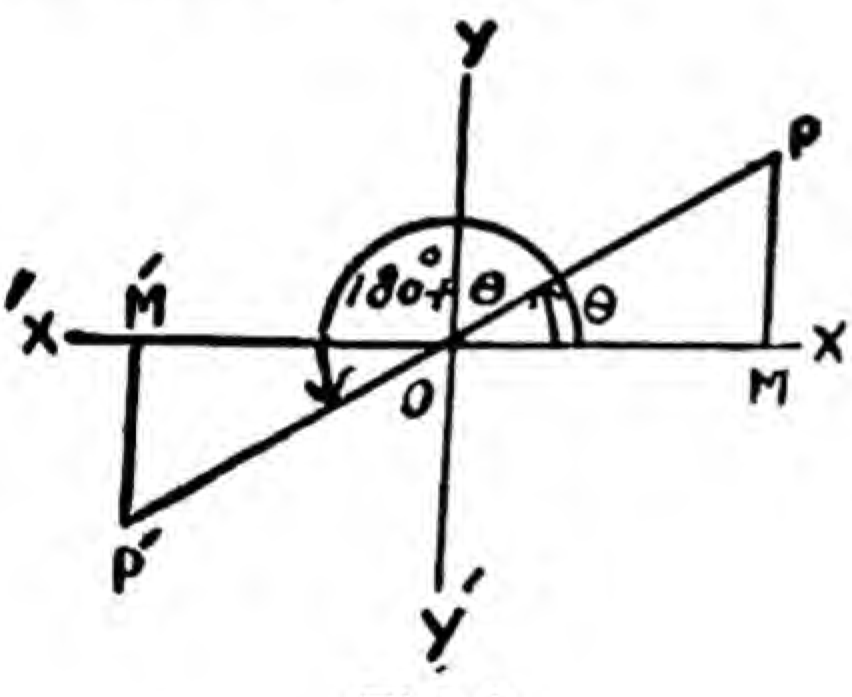


Fig. 1

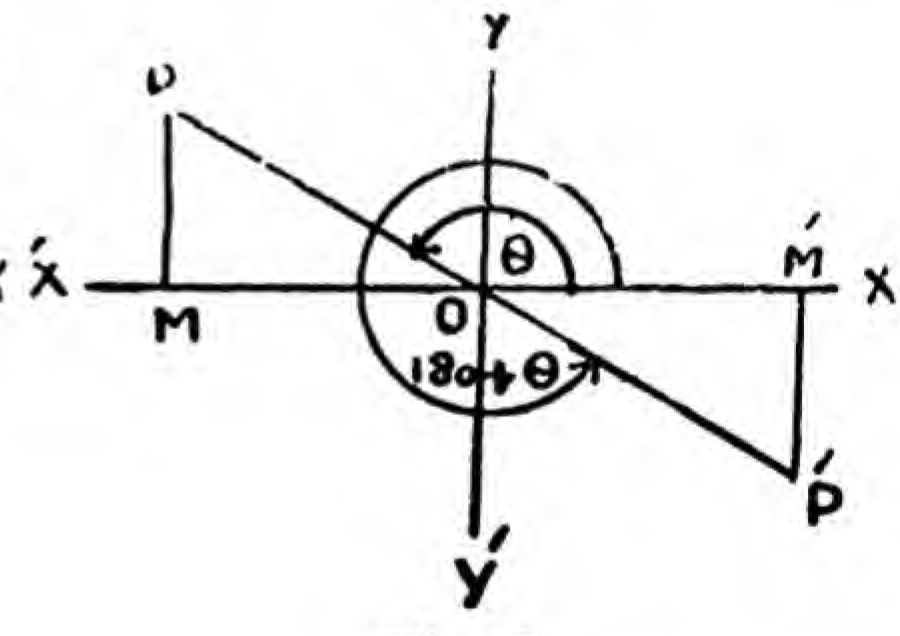


Fig. 2

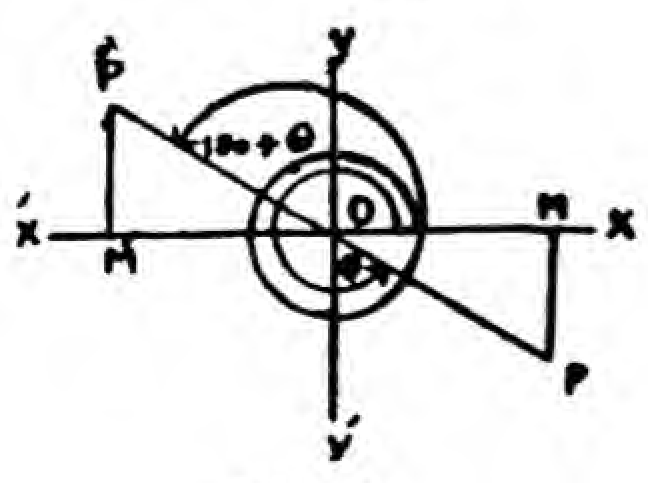


Fig. 3

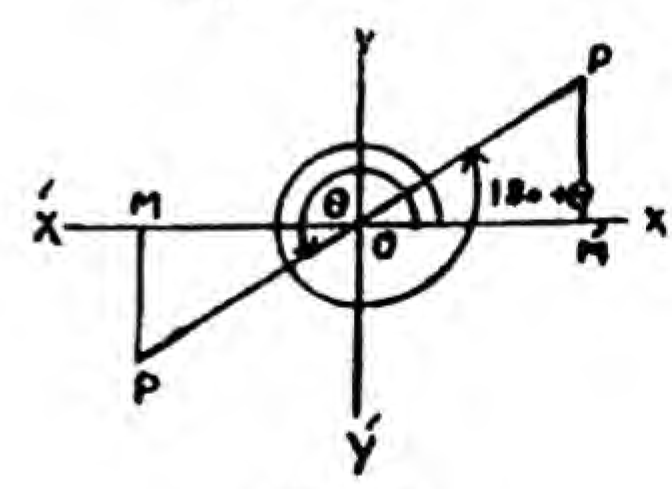


Fig. 4

Draw PM, P'M' $\perp$s to XOX'.

Then the $\triangle$s OMP and OM'P' are congruent. Therefore having regard to the signs of the lines, we have

$$OM' = -OM$$
$$M'P' = -MP$$
$$OP' = OP$$

$$\therefore \sin(180+\theta) = \frac{M'P'}{OP'} = \frac{-MP}{OP} = -\sin\theta$$

$$\cos(180^\circ+\theta) = \frac{OM'}{OP'} = \frac{-OM}{OP} = -\cos\theta$$

$$\tan(180^\circ+\theta) = \frac{M'P'}{OM'} = \frac{-MP}{-OM} = \frac{MP}{OM} = \tan\theta$$

$$\cot(180+\theta) = \frac{OM'}{M'P'} = \frac{-OM}{-MP} = \frac{OM}{MP} = \cot\theta$$

$$\sec(180^\circ+\theta) = \frac{OP'}{OM'} = \frac{OP}{-OM} = -\sec\theta$$

$$\operatorname{cosec}(180+\theta) = \frac{OP'}{M'P'} = \frac{OP}{-MP} = -\operatorname{cosec}\theta$$

Rule : The results of the above 4 articles can be briefly stated as :—

When an angle θ is added to or subtracted from an even multiple of a right angle (i. e. 180°, 360°, *etc.) there is no change in the form of the trigonometrical ratios. The sign to be attached to the T-ratios is determined by the rule of "All-sin-tan-cos" depending on the quadrant in which the revolving line will lie for the angles* $180^\circ+\theta$, $360^\circ+\theta$, *etc., regarding θ to be acute.*

Cor. 1. (*i*) $\sin 180^\circ = \sin(180^\circ+0^\circ) = -\sin 0^\circ = 0$

(*ii*) $\cos 180^\circ = \cos(180^\circ+0^\circ) = -\cos 0^\circ = -1$

Cor. 2. (*i*) $\sin 270^\circ = \sin(180^\circ+90^\circ) = -\sin 90^\circ = -1$

(*ii*) $\cos 270^\circ = \cos(180^\circ+90^\circ) = -\cos 90^\circ = 0$

5. To find the T-ratios of $90°-\theta$, in terms of those of θ for all values of θ.

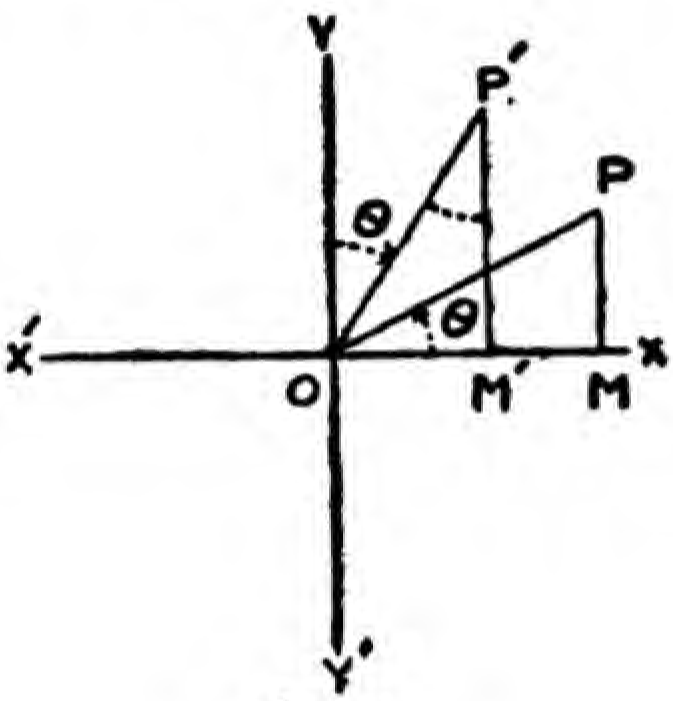


FIG. 1

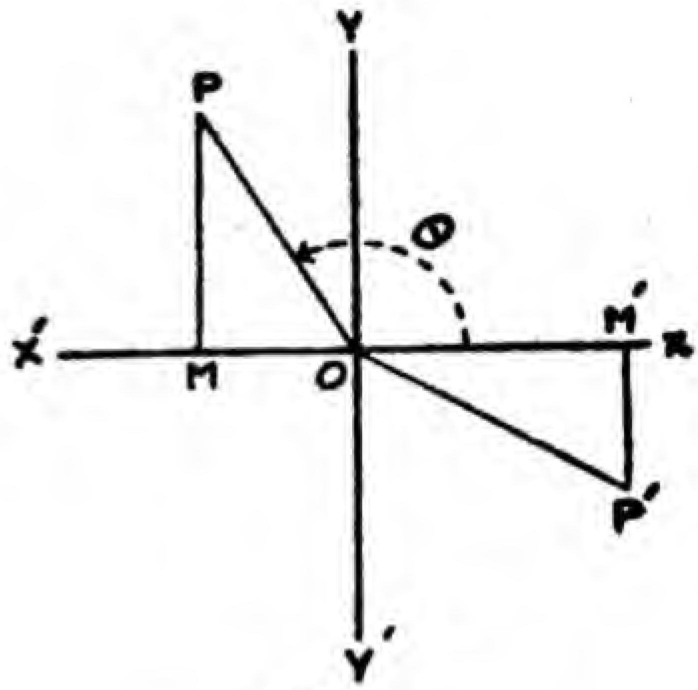


FIG. 2

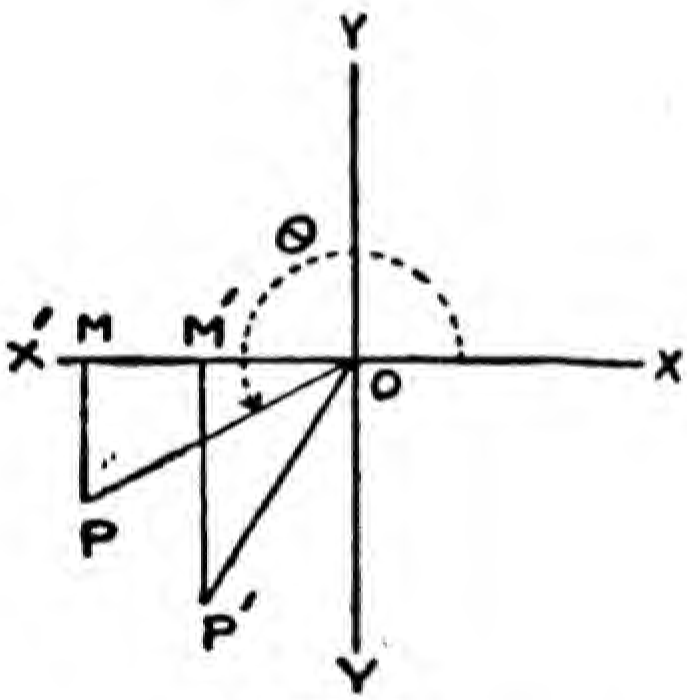


FIG 3

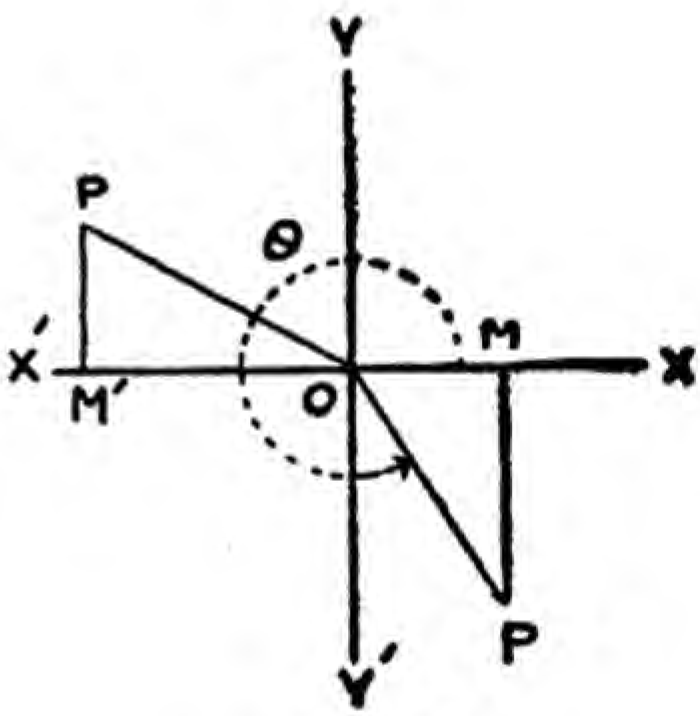


FIG. 4

Let a revolving line OP, starting from OX, revolve through $\angle XOP=\theta$, lying in any quadrant.

Let another revolving line OP′ (=OP) starting from OX revolve through 90° and then revolve back through θ so that $\angle XOP' = 90° - \theta$.

Draw PM, P′M′ $\perp$s to XOX′

Then in $\triangle$s OMP and OM′P′,

$OP = OP'$, $\angle MOP = M'P'O$,

$\angle OMP = \angle O'M'P' = 1$ rt. $\angle$.

$\therefore$ $\triangle$s are congruent. Therefore having regard to the signs of the lines we have,

$OM' = MP$

$M'P' = OM$

$OP' = OP$

$$\therefore \sin(90° - \theta) = \frac{M'P'}{OP'} = \frac{OM}{OP} = \cos\theta$$

$$\cos(90° - \theta) = \frac{OM'}{OP'} = \frac{MP}{OP} = \sin\theta$$

$$\tan(90° - \theta) = \frac{M'P'}{OM'} = \frac{OM}{MP} = \cot\theta$$

$$\cot(90° - \theta) = \frac{OM'}{M'P'} = \frac{MP}{OM} = \tan\theta$$

$$\sec(90° - \theta) = \frac{OP'}{OM'} = \frac{OP}{MP} = \operatorname{cosec}\theta$$

$$\operatorname{cosec}(90° - \theta) = \frac{OP'}{M'P'} = \frac{OP}{OM} = \sec\theta.$$

6. To find the T-ratios of $90°+\theta$ in terms of those of θ for all values of θ.

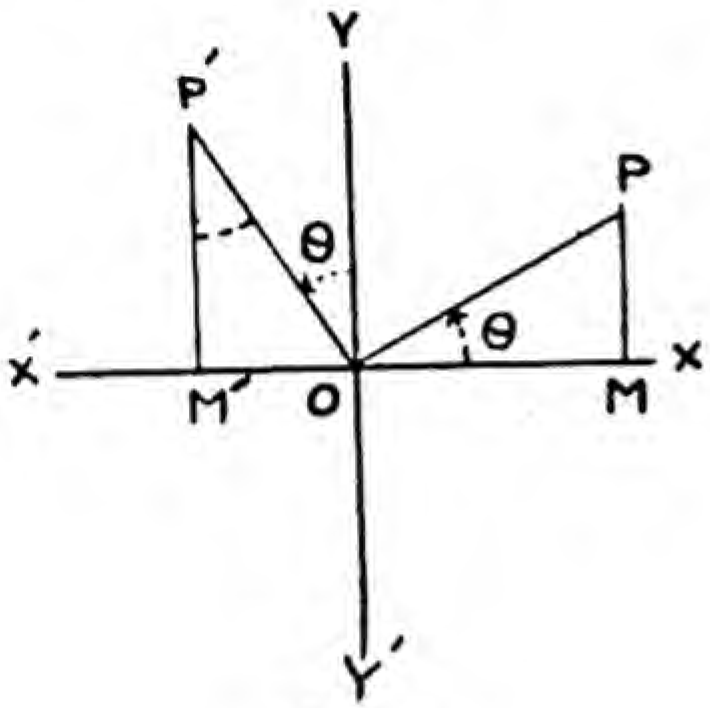


FIG. 1

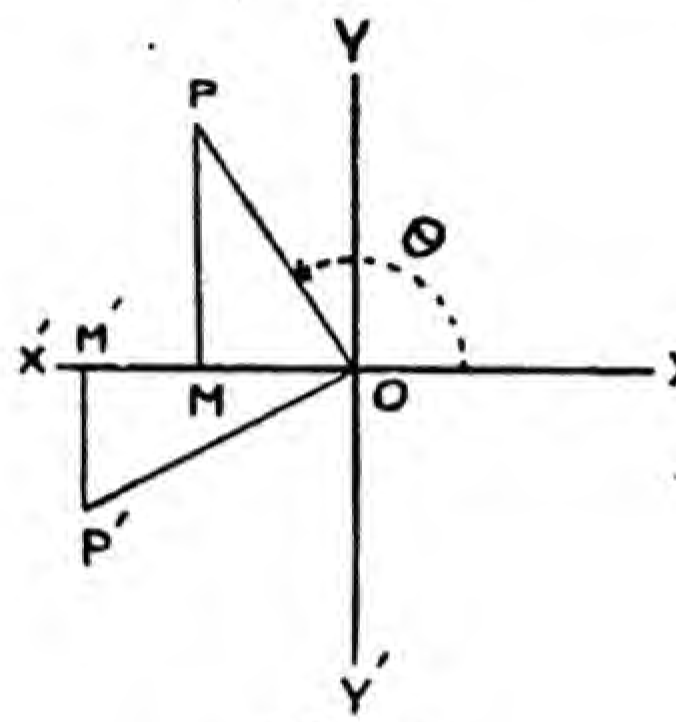


FIG. 2

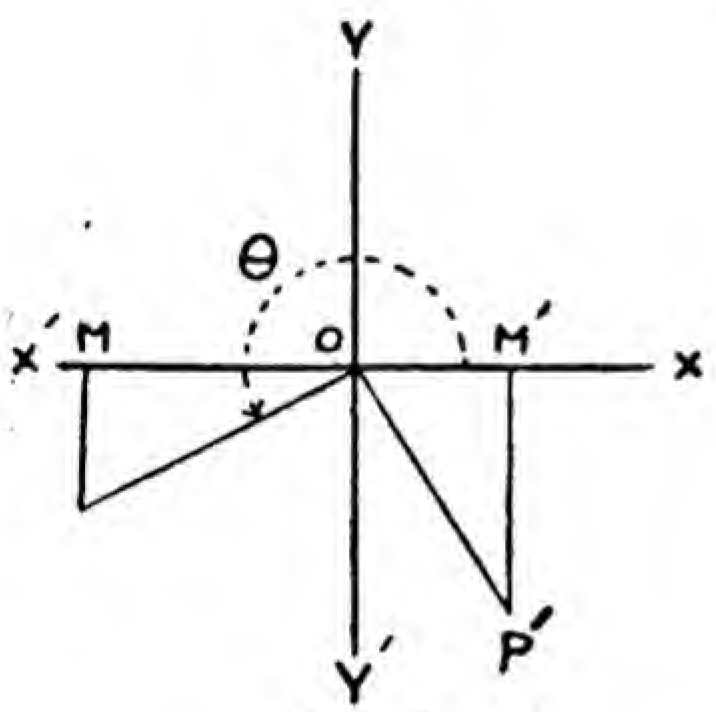


FIG 3

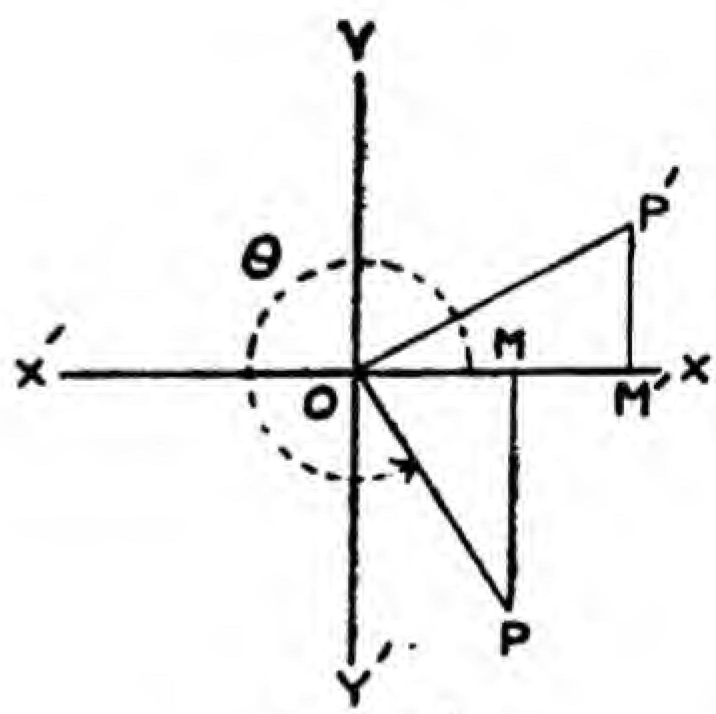


FIG. 4

Let the revolviug line. starting from OX, revolve through $\angle XOP=\theta$, lying in any quadrant.

Let another revolving line OP' (=OP), starting from OX trace out $90°$ and then revolve further through θ, so that $\angle XOP'=90°+\theta$.

Draw PM, P'M' $\perp$s to XOX'.
Then in $\triangle$s OMP and OM'P',
$OP=OP'$, $\angle MOP=M'P'O$
$\angle OMP=OM'P'=1\text{rt.}\angle$.

$\therefore$ $\triangle$s are congruent. Therefore, having regard to the signs of the lines we have, $OM' = -MP$

$$M'P' = OM$$

$$OP' = OP$$

$$\therefore \sin(90^\circ+\theta) = \frac{M'P'}{OP'} = \frac{OM}{OP} = \cos\theta$$

$$\cos(90^\circ+\theta) = \frac{OM'}{OP'} = \frac{-MP}{OP} = -\sin\theta$$

$$\tan(90^\circ+\theta) = \frac{M'P'}{OM'} = \frac{OM}{-MP} = -\cot\theta$$

$$\cot(90^\circ+\theta) = \frac{OM'}{MP'} = \frac{-MP}{OM} = -\tan\theta$$

$$\sec(90^\circ+\theta) = \frac{OP'}{OM'} = \frac{OP}{-MP} = -\operatorname{cosec}\theta$$

$$\operatorname{cosec}(90^\circ+\theta) = \frac{OP'}{M'P'} = \frac{OP}{OM} = \sec\theta.$$

Note. Similarly we can find T-ratios of $270^\circ-\theta$ and $(270+\theta)$ in terms of those of θ.

The results of **Arts.** **5**, 6 can be briefly stated as :—

Rule. *When an angle θ is added to or subtracted from an odd multiple of a right angle* (*i. e.* 90°, 270° *etc.*) *the T-ratios are changed into co-ratios and vice versa* (*i. e. sine into cosine, tangent into cotangent etc*) *The sign to be attached to the T-ratio is given by the rule "All—sin—tan—cos", depending on the quadrant in which the revolving line lies for angles* $90^\circ \pm \theta$, $270^\circ \pm \theta$, *regarding θ to be acute.*

7. Function of θ. The value of each of $\sin\theta$, $\cos\theta$, $\tan\theta$ etc. depends upon the value of θ and varies with the change in the value of θ; hence $\sin\theta$, $\cos\theta$, $\tan\theta$, etc. are called **Functions of** θ and writtten as $f(\theta)$; while θ is called the variable.

Periodic function. If k is the least positive constant such that when θ is changed to $\theta+k$, the value of a function of θ remains unchanged, the function is called **Periodic** and k is known as the **Period** of the function.

$\because \sin(\theta+2\pi)=\sin(\theta+4\pi)=\ldots=\sin(\theta+2n\pi)$

and $\cos\theta=\cos(\theta+2\pi)=\cos(\theta+4\pi)=\ldots=\cos(\theta+2n\pi)$

$\therefore$ $\sin\theta$ and $\cos\theta$ are periodic functions of period 2π.

Similarlp sec θ and cosec θ are periodic functions of period 2π.

Also, $\because \tan\theta=\tan(\theta+\pi)=\tan(\theta+2\pi)$
$=\ldots\ldots=\tan(\theta+n\pi)$

$\therefore$ $\tan\theta$ and $\cot\theta$ are periodic functions of period π.

Ex. 1. Find the values of

(*i*) $\cos 480°$, (*ii*) $\tan 540°$, (*iii*) $\sin(1305°)$

(*iv*) $\operatorname{cosec}(-960°)$.

(*i*) $\cos 480°=\cos(360°+120°)=\cos 120°$
$=\cos(180°-60°)=-\cos 60°=-\frac{1}{2}$

(*ii*) $\tan 540°=\tan(360+180°)=\tan 180°$
$=\tan(180+0°)=\tan 0°=0$

(*iii*) $\sin 1305°=\sin(3\times 360°+225°)=\sin 225°$

$$=\sin(180°+45°)=-\sin 45°=-\frac{1}{\sqrt{2}}$$

(*iv*) $\operatorname{cosec}(-960°)=-\operatorname{cosec} 960°$

$=-\operatorname{cosec}(2\times 360°+240°)=-\operatorname{cosec} 240°$

$$=-\operatorname{cosec}(180°+60°)=+\operatorname{cosec} 60°=\frac{2}{\sqrt{3}}$$

Ex. 2. In a $\triangle ABC$, show that

$$\sin B=\sin(A+C) \text{ and } \sin\frac{B}{2}=\cos\frac{A+C}{2}$$

$\because A+B+C=\pi$, $\therefore B=\pi-(A+C)$

$\therefore$ $B=(\pi-\overline{A+C})=\sin(A+C)$

Also $\because \dfrac{A+B+C}{2}=\dfrac{\pi}{2}$, $\therefore \dfrac{B}{2}=\dfrac{\pi}{2}-\dfrac{A+C}{2}$

$$\therefore \sin\frac{B}{2}=\sin\left(\frac{\pi}{2}-\frac{A+C}{2}\right)=\cos\frac{A+C}{2}$$

Ex. 3. Prove that

$$\sin 120^\circ \sin 780^\circ + \cos 120^\circ \cos 420^\circ = \tfrac{1}{2}$$

Sol. L.H.S. $= \sin 120^\circ \sin 780^\circ + \cos 120^\circ \cos 420^\circ$

$$= \sin(180^\circ - 60^\circ)\sin(2\times 360^\circ + 60^\circ) + \cos(180^\circ - 60^\circ)\cos(360^\circ + 60^\circ)$$

$$= \sin 60^\circ \times \sin 60^\circ - \cos 60^\circ \cos 60^\circ$$

$$= \sin^2 60^\circ - \cos^2 60^\circ$$

$$= \left(\frac{\sqrt{3}}{2}\right)^2 - \left(\tfrac{1}{2}\right)^2$$

$$= \frac{3}{4} - \frac{1}{4}$$

$$= \frac{1}{2}.$$

Ex. 4. Simplify :

$$\frac{\sin(90^\circ - \theta)}{\sin(90^\circ + \theta)} - \frac{\tan(180^\circ + \theta)}{\cos(90^\circ + \theta)} + \frac{\sin(180 - \theta)}{\cot(360^\circ - \theta)\sin^2(-\theta)}$$

Sol.
$$\frac{\sin(90^\circ - \theta)}{\sin(90^\circ + \theta)} - \frac{\tan(180^\circ + \theta)}{\cos(90^\circ + \theta)} + \frac{\sin(180^\circ - \theta)}{\cot(360^\circ - \theta)\sin^2(-\theta)}$$

$$= \frac{\cos\theta}{\cos\theta} - \frac{\tan\theta}{-\sin\theta} + \frac{\sin\theta}{-\cot\theta\sin^2\theta}$$

$$= 1 + \frac{\sin\theta}{\cos\theta}\cdot\frac{1}{\sin\theta} - \frac{\sin\theta}{\frac{\cos\theta}{\sin\theta}\cdot\sin^2\theta}$$

$$= 1 + \frac{1}{\cos\theta} - \frac{1}{\cos\theta}$$

$$= \mathbf{1}.$$

Exercise 5.

1. Find the values of $\cos 225°$, $\sin 4620°$, $\tan(-585°)$, $\sec\dfrac{10\pi}{3}$. (P. U.)

Prove that :—

2. $\text{Sin } 780° \sin 120° + \cos 120° \sin 390° = \frac{1}{2}$
3. $\text{Sin } 600° \cos 330° + \cos 120° \sin 150° = -1.$
4. $\text{Sin } (270°+A) \operatorname{cosec}(-A) + \tan(270°+A) = 0.$
5. $\text{Sin}^2 36° - \sin^2 18° = \sin^2 72° - \sin^2 54°.$
6. $\text{Sin}\left(\frac{\pi}{4}+A\right) = \cos\left(\frac{\pi}{4}-A\right)$
7. Simplify the following :—

(*i*) $\dfrac{\cos(90°+\theta)\sec(-\theta)\tan(180°-\theta)}{\sec(360°-\theta)\sin(180°+\theta)\cot(90°-\theta)}$

(*ii*) $\dfrac{\sin(180°-\theta)\cos(270°-\theta)}{\sin(180°-\theta)\cos(270°+\theta)}$

8. Prove that

$$\frac{\cos\theta}{\sin(90°+\theta)} + \frac{\sin(-\theta)}{\sin(180°+\theta)} - \frac{\tan(90°+\theta)}{\cot\theta} = 3$$

(D. U. 1951)

9. Prove that $\dfrac{\sec A + \tan(180°-A)}{\sec A - \tan(180°-A)} - \dfrac{\tan A - \sec(180°-A)}{\tan A + \sec(180°-A)}$
$= 2 + 4\tan^2 A$ (J. & K. U. 1954)

10. In a $\triangle$ ABC, prove taht (*i*) $\sin\dfrac{A}{2} = \cos\dfrac{B+C}{2}$

(*ii*) $\cos\dfrac{A}{2} = \sin\dfrac{B+C}{2}$

11. In a $\triangle$ ABC, show that $\cos A = -\cos(B+C)$
12. In a cyclic quadrilateral prove that (*i*) $\sin A = \sin C$ and (*ii*) $\cos B + \cos D = 0$. (J. & K. U. 1958)
13. Find all the values of θ lying between 0° and 360° for

which :—

(*i*) $\cos\theta=\frac{\sqrt{3}}{2}$ (*ii*) $\sin\theta=\frac{-\sqrt{3}}{2}$,

(*iii*) $\sec\theta=2$. (*iv*) $\operatorname{cosec}\theta=\sqrt{2}$. (*v*) $\tan\theta=-1$.

14. Define a periodic function. What is the period of $\tan\theta$? (P. U. 1942)

15. (*i*) Prove that for all values of θ, $\tan(\pi+\theta)=\tan\theta$. (J. & K. U. 1957)

(*ii*) Prove „ „ „ „ θ, $\sin(90°+\theta)=\cos\theta$ (J. & K. U. 1951)

16. Define the secant of an angle θ.

Prove taht (*i*) $\sec(-\theta)=\sec\theta$. (J. & K. U. 1953)

(*ii*) $\sin(n\pi+\theta)=(-1)^n\sin\theta$. (P.U.1951)
where n is a positive integer,

17. By drawing figures in several quadrants, prove that $\tan\left(\frac{\pi}{2}+\theta\right)=-\cot\theta$ (J. & K. U. 1956)

18. Prove that $\tan\theta\tan\left(\frac{\pi}{2}\pm\theta\right)\pm1=0$ (J.&K. U. 1957)

———

CHAPTER IV.

Variations of Trigonometrical ratios and their graphs.

1. To trace the variations of sin θ as θ varies from 0° to 360°.

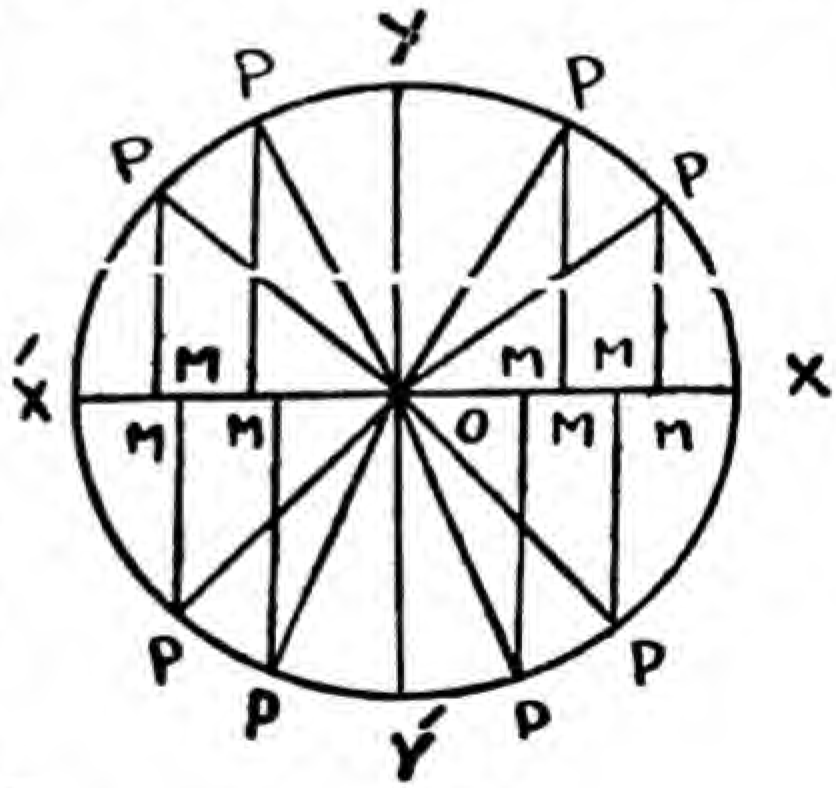
With O as centre draw a circle of unit radius. Through O draw the diameters XOX′ and YOY′ at right angles and take OX as the initial line. Let the revolving line OP, equal to unity in length, trace out $\angle XOP=\theta$, of any magnitude and draw PM $\perp$XOX′, then

$$\sin\theta=\frac{MP}{OP}=\frac{MP}{1}=MP.$$

Hence variations in $\sin\theta$ depend on the variations in the values of MP.

First Quadrant. As θ increases from 0° to 90°, MP is positive and increases from 0 to OY.

∴ sin θ is positive and varies from 0 to 1.



Second Quadrant. As θ increases from 90° to 180°, MP is positive and decreases from OY to 0.

∴ sin θ is positive and varies from 1 to 0.

Third Quadrant. As θ increases from 180° to 270°,

MP is negative and increases in magnitude from 0 to OY′.

∴ sin θ is negative and varies from 0 to—1.

Fourth Quadrant. As θ increases from 270° to 360°, MP is negative and decreases in magnitude from OY′ to 0.

∴ sin θ is negative and varies from—1 to 0.

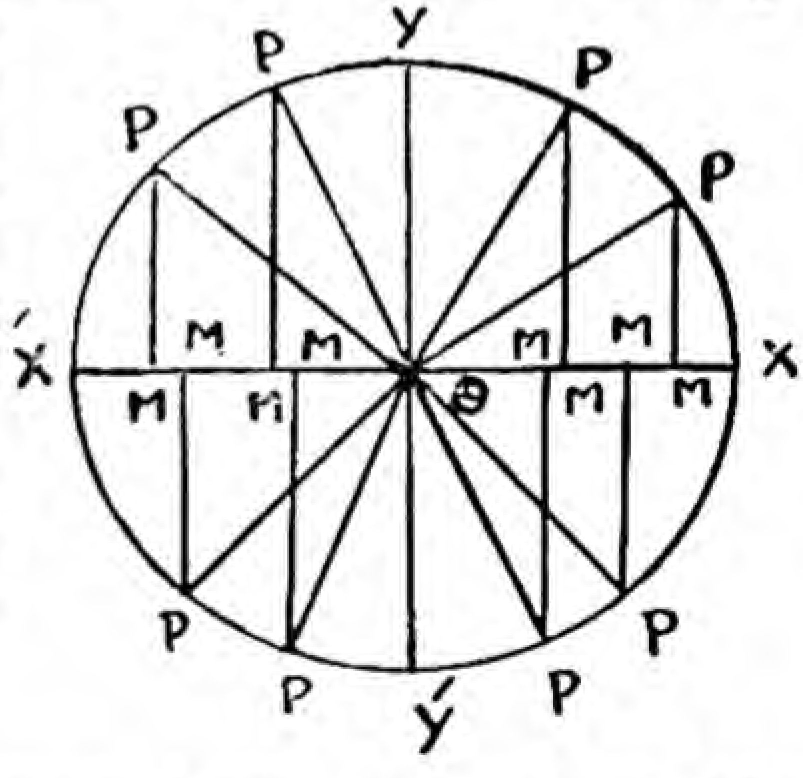
Note 1. It follows that sin θ is never > 1 numerically and can have values between 1 and—1.

It also follows that there are two angles between 0° and 360° which have a given sine. If the given sine is positive, the angles lie between 0° and 180°. If the given sine is negative the angles lie between 180° and 360°.

Note. 2 When MP is negative we use the words 'increases or decreases in magnitude' for if a negative quantity increases in magnitude it decreases algebraically and if it decreases in magnitude it increases algebraically.

2. To find the variations of cos θ as θ varies from 0° to 360°.

With O as centre draw a circle of unit radius. Through O draw the diameters XOX' and YOY' at right angles and take OX as the initial line. Let the revolving line OP, equal to unity in length, trace out $\angle XOP=\theta$ of any magnitude and draw $PM \perp X'OX$, then,



$$\cos\theta=\frac{OM}{OP}=\frac{OM}{1}=OM.$$

Hence variations in cos θ depend on the variations in the value of OM.

First Quadrant. As θ increases from 0° to 90°, MO is positive and decreases from 1 to 0.

∴ cos θ is positive and varies from 1 to 0.

Second Quadrant. As θ increases from 90° to 180°, OM is negative and decreases in magnitude from 0 to OX'.

∴ cos θ is negative and varies from 0 to—1.

Third Quadrant. As θ increases from 180° to 270°, OM is negative and decreases in magnitude from OX' to 0.

∴ cos θ is negative and varies from—1 to 0.

Fourth Quadrant. As θ increases from 270° to 360°, OM is positive and increases from 0 to OX.

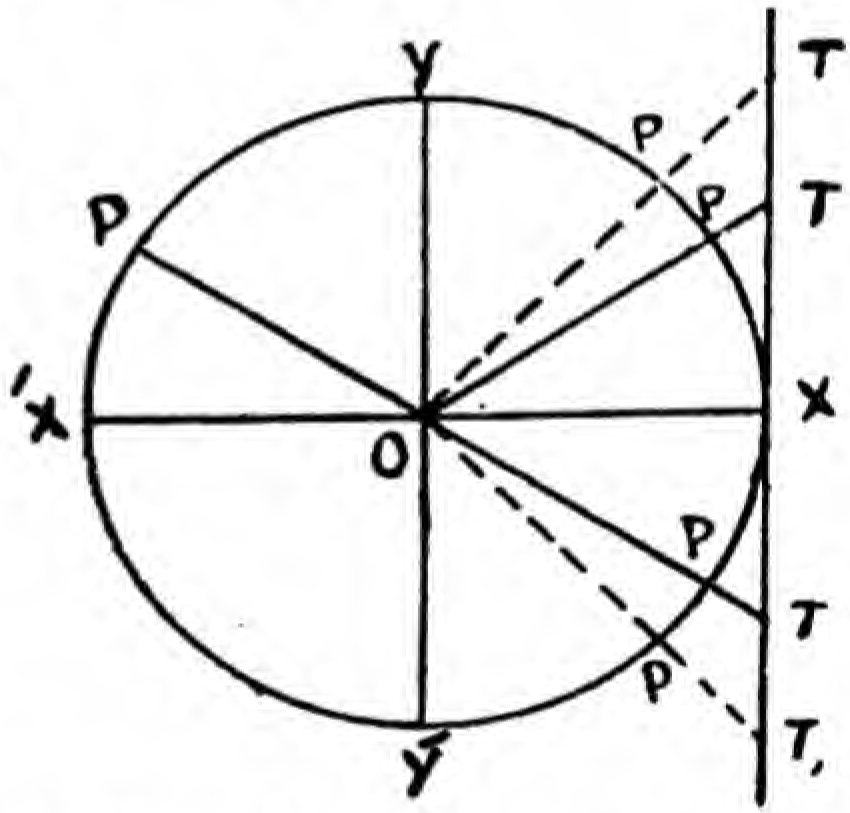
∴ cos θ is positive and varies from 0 to 1.

Note. It follows that $\cos\theta$ is never $>$ 1 and can have any value between 1 and—1.

It also follows that there are two angles lying between 0° and 360°, which have a given cosine, if the given cosine is positive one angle lies between 0° and 90° and the other between 270° and 360°. But if the given cosine is negative then the two angles lie between 90° and 270°.

3. To trace the variations of tan θ as θ varies from 0° to 360°.

With O as centre draw a circle of unit radius.



Through O draw the diameters XOX′ and YOY′ at right angles to each other and take OX as intial line. Let the revolving line OP, equal to unity in length, trace out $\angle XOP=\theta$ of any magnitude. At X draw a tangent to the circle and produce OP to meet it in T.

In $\triangle XOT$,

$$\tan\theta=\frac{XT}{OX}=\frac{XT}{1}=XT.$$

Hence variations in tan θ depend on the variations in the value of XT.

Remembering that XT is positive when drawn above and negative if drawn below OX, we have ;

First Quadıant When $\theta=0°$, $XT=0$.

As θ increases, XT is positive and increases. When $\theta=90°$, OP becomes parallal to XT (*i. e.* meets XT at $\propto$). ∴ as $\theta\to 90°$, $XT\to\infty$.

∴ as θ increases from 0° to 90°,

tan θ is positive and varies from 0 to ∞.

Crossing 90°. When θ is a little $<90°$, XT is positive and very large. When θ is a little $>90°$, T lies below OX and therefore XT is negative and very large.

$\therefore$ **as θ crosses 90°, tan θ suddenly changes from $+\infty$ to $-\infty$.**

Second Quadrant. As θ increases from 90° to 180°,
XT is negative and decreases in magnitude.
At $\theta=180°$, XT$=0$
$\therefore$ **tan θ is negative and varies from $-\infty$ to 0,**
Third Quadrant. As θ increases from 180° to 270°,
XT is positive and increases.
when $\theta=270°$, OP$\parallel$XT.
$\therefore$ **tan θ is positive and varies from 0 to ∞.**

Crossing 270°. As θ passes through 270°, tan θ suddenly changes from $+\infty$ to $-\infty$ (as it does in passing through 90°).

4th Quadrant. As θ increases from 270° to 360°,
XT is negative and decreases in magnitude.
At $\theta=360°$, XT$=0$.

$\therefore$ **tan θ is negative and varies from $-\infty$ to 0.**

4. To trace the variations of sec θ as θ increases from 0° to 360°.

Making use of the fig. of Art. 3, we have

$$\text{from } \triangle\, \text{OXT, sec } \theta=\frac{\text{OT}}{\text{OX}}=\frac{\text{OT}}{1}=\text{OT}.$$

$\therefore$ the variations of sec θ are the same as those of OT.

Remembering that OT is positive if drawn along OP and negative if drawn along PO produced, we have :—

First Quadrant. When $\theta=0$, OT$=$OX$=1$,

As θ increases, OT is positive and increases.

When $\theta=90°$, OP$\parallel$XT (*i.e.* meets XT at ∞)

$\therefore$ As θ increases from 0° to 90°,

Sec θ is positive and increases from 1 to ∞.

Crossing 90°. When θ is a little $<90°$, OT is positive and very large ;

When θ is a little $>90°$, OT is negative and very large.

$\therefore$ as θ passes through 90°, sec θ changes from $+\infty$ to $-\infty$.

Second Quadrant As θ increases from 90° to 180°,
OT is negative and decreases in magnitude,
At $\theta=180°$, OT$=$OX$=-$OP$=-1$
$\therefore$ **Sec θ is negative and varies from $-\infty$ to -1.**

Third Quadrant As θ increases from 180° to 270°,
OT is negative and increases in magnitude.

At $\theta=270°$, OT coincides with OY and is $\parallel$XT (*i. e.* meets XT at $-\infty$)

$\therefore$ **Sec θ is negative and varies from ∞ to 1.**

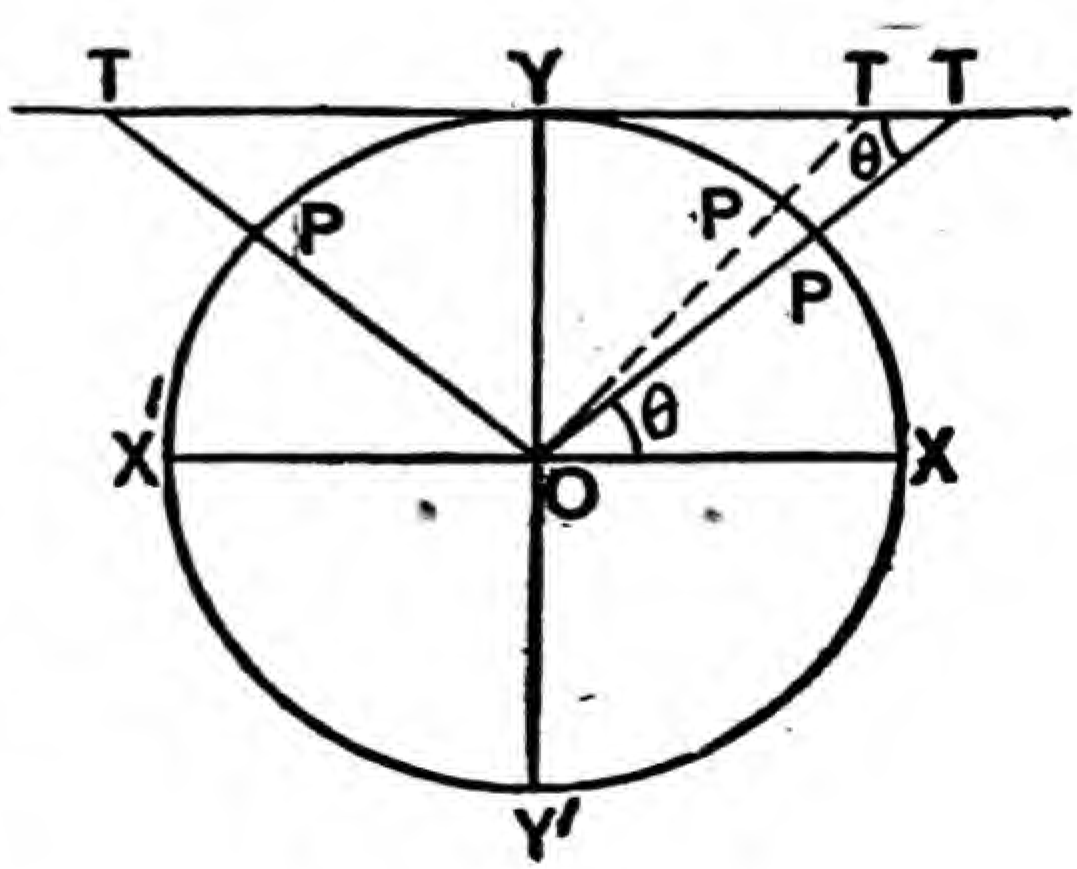
Crossing 270°. sec θ changes from $-\infty$ to $+\infty$ (as it does at $\theta=90°$).

Fourth Quadrant As θ increases from 270° to 360°,
OT is positive and decreases.
At $\theta=360°$, OT$=$OX$=1$
$\therefore$ **Sec θ is positive and varies from ∞ to 1.**

Note 1. As sec θ varies from 1 to ∞ and from $-\infty$ to -1, sec θ is always >1 numerically.

Note 2. We have noticed above that when a T-ratio becomes infinite for some value of θ, it changes its sign as the angle passes through that value.

5. **To trace the variations of cot θ as θ increases from 0° to 360°.**



With O as centre draw a circle of unit radius. Through O draw the diameters XOX′ and YOY′ at right angles to each other and take OX as the initial line Let OP trace out $\angle$XOP$=\theta$ of any magnitude. At Y draw a tangent to the circle and produce OP to meet it in T.

Now cot $\theta=$cot XOP
$=$cot OTY

$$=\frac{YT}{OY}=\frac{YT}{1}=YT.$$

Hence variations in cot θ depend on variations in the value of YT.

Remembering that YT is positive when T is to the right of OY and negative if T is to the left of OY, we have :

First Quadrant. When $\theta=0$, OP‖YT, ∴ $YT=\infty$.

As θ increases, YT is positive and decreases.

When $\theta=90°$, $YT=0$.

∴ as θ increases from 0° to 90°,
cot θ is positive and varies from ∞ to 0.
Second Quadrant. As θ increases from 90° to 180°.
YT is negative and increases in magnitude,
At $\theta=180°$, OP‖YT and ∴ $YT=-\infty$.
∴ **cot θ varies from 0 to $-\infty$.**

Crossing 180°. When θ is a little <180°, YT is negative and very large.

When θ is a little >180°, YT is positive and very large.

∴ as θ passes through 180°, cot θ changes from $-\infty$ to $+\infty$.

Third Quadrant. As θ increases from 180° to 270°,
YT is positive and decreases.
At $\theta=270°$, $YT=0$
∴ **cot θ is positive and varies from ∞ to 0.**
Fourth Quadrant. As θ increases from 270° to 360°,
YT is negative and increases in magnitude.
At $\theta=360°$, OP‖YT, ∴ $YT=-\infty$

∴ **cot θ is negative and varies from 0 to $-\infty$**

Note. Cot θ can take any value, for it varies from ∞ to 0 and from 0 to $-\infty$.

6. To trace the variations of cosec θ as θ increases from 0° to 360°.

Making use of the fig. in art. 5, we have :

$$\text{Cosec } \theta=\text{cosec XOP}=\text{cosec YTO}=\frac{OT}{OY}=OT$$

$\therefore$ the variations of cosec θ are the same as those of OT.

Remembering that OT is positive if drawn along OP and negative if drawn along PO produced we have:

First Quadrant. At $\theta=0$, OT$\|$YT, $\therefore$ OT$=\infty$

As θ increases, OT is positive and decreases.

At $\theta=90°$, OT$=$OY$=1$.

$\therefore$ as θ increases from 0° to 90°,

Cosec θ varies from ∞ to 1.

Second Quadrant. As θ increases from 90° to 180°,

OT is positive and increases.

At $\theta=180°$, OT$\|$YT, $\therefore$ OT$=\infty$,

$\therefore$ **Cosec θ varies from 1 to ∞.**

Crossing 180°. When θ is a little $<180°$, OT is positive and very large;

When θ is a little $>180°$, OT is negative and very large;

$\therefore$ as θ passes through 180°, cosec θ changes from $+\infty$ to $-\infty$.

Third Quadrant As θ increases from 180° to 270°,

OT is negative and decreases in magnitude.

At $\theta=270°$, OT$=$OY$=-1$.

$\therefore$ **Cosec θ varies from $-\infty$ to -1.**

Fourth Quadrant. As θ increases from 270° to 360°,

OT is negative and increases in magnitude.

At $\theta=360°$, OT$\|$YT, $\therefore$ OT$=--\infty$.

$\therefore$ **Cosec θ is negative and varies from -1 to $-\infty$.**

Note. As cosec θ varies from ∞ to 1 and from $-\infty$ to -1 therefore it is always $\geqslant$ 1 numerically.

Graphs.

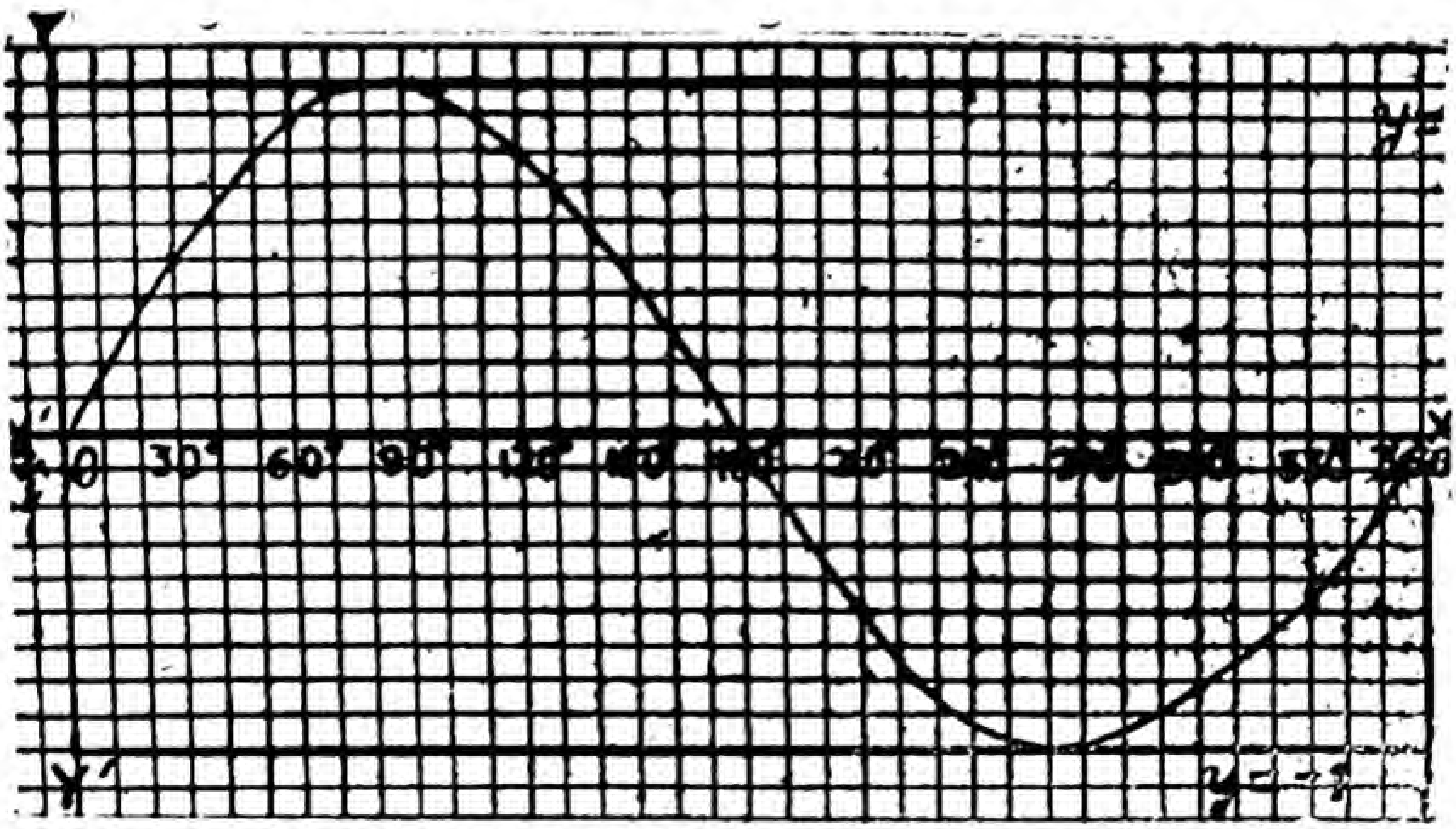
7. To draw the graph of sin θ as θ changes from 0° to 360°.

(*i*) Put $y = \sin\theta$. Tabulating values we have :—

θ	0°	30°	60°	90°	120°	150°	180°	210°	240°	270°	300°	330°	360°
y or sin θ	0	·5	·87	1	·87	·5	0	−·5	−·87	−1	−·87	−·5	0

(*ii*) Draw two lines OX and OY at right angles to each other on the graph paper. Let each small division along OX represent 10° and each small division along OY represent 1. Measure θ in terms of degrees along OX and corresponding values of sin θ along OY.

(*iii*) Plot the points given by the above table and draw a smooth curve with a free hand through these points as shown below.



Cor. Read the value of sin 50° from the graph. Measure along OX 50° (=5 small divisions). At the fifth division, draw a line ⊥ OX; where this line cuts the graph read that ordinate, that will give the value of sin 50°.

Note. 1. The 3 steps noted above are necessary in all the graphs.

Note. 2. If the graph of sin θ is required between 0° and −360°, the table of values can be obtained from the previous table with the help of the formula $\sin(-\theta) = -\sin\theta$ between 0° and −360°. Thus the table will be :—

θ	0°	−30°	−60°	−90°	−120°	−150°	−180°	−210°	−240°	−270°	−300°	−330°	−360°
sin θ	0	−·5	−·87	−1	−·87	−·5	0	·5	·87	1	·87	·5	0

8. To draw the graph of cos θ as θ varies from 0° to 360°

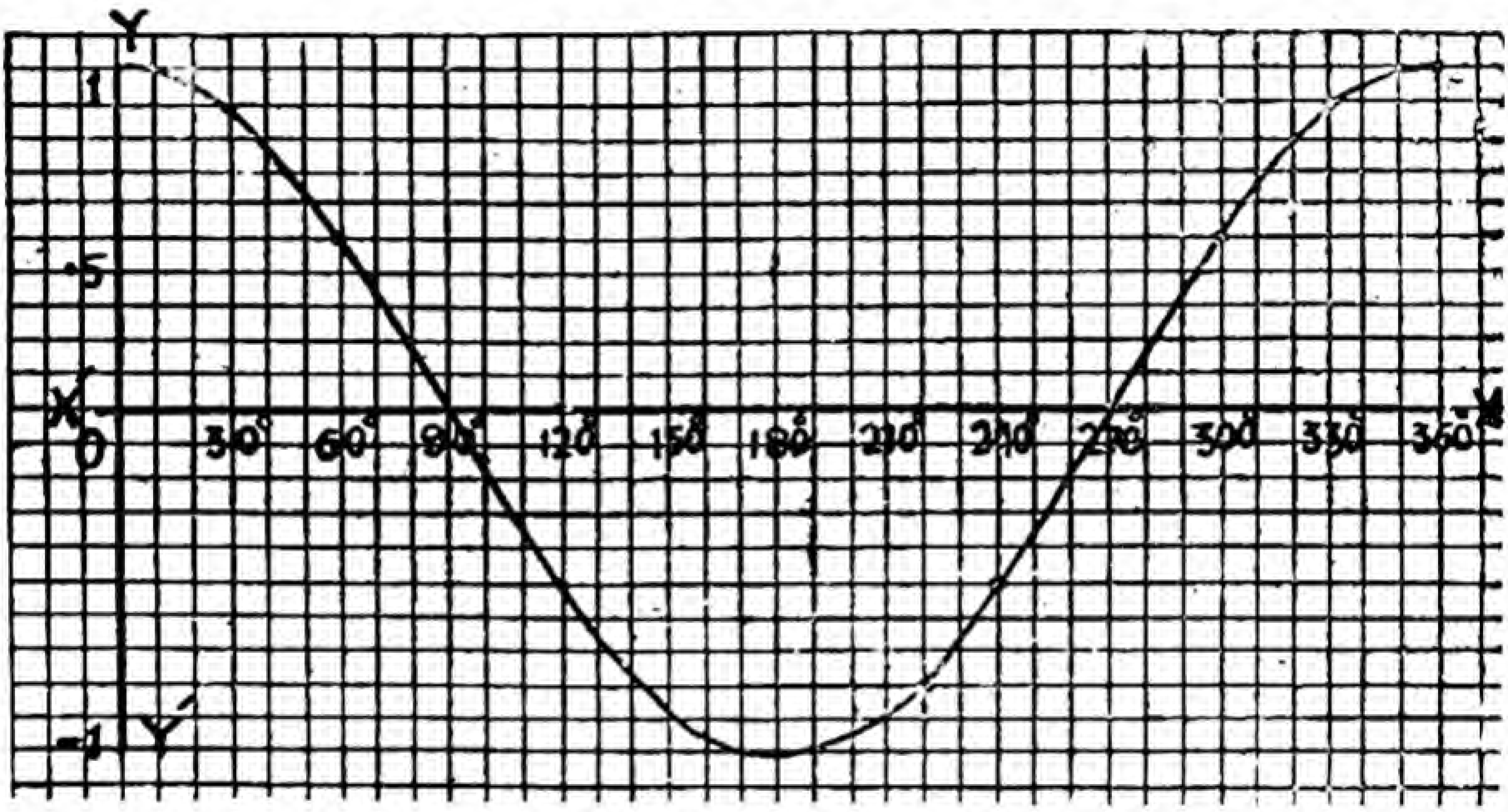
(*i*) Put $y = \cos\theta$. Tabulating values we have :—

θ	0°	30°	60°	90°	120°	150°	180°	210°	240°	270°	300°	330°	360°
y or cos θ	1	·87	·5	0	−·5	−·87	−1	−·87	−·5	0	·5	·87	1

(*ii*) Let each small division along OX $=10^\circ$

and let ,, ,, ,, ,, OY $=\cdot 1$

(*iii*) Plot points given by the above table and draw a smooth curve with a free hand through these as shown below.



Cor. From the graph of cosine θ find the value of θ such that $\cos\theta=\frac{3}{5}$.

since $\cdot 1=1$ small division, $\therefore\ \frac{3}{5}=\cdot 6=6$ small divisions.

Measure along OY 6 small divisions, and at the sixth division, draw a line $\perp$OY. Where this line meets the graph, read the abscissae. These are values of θ.

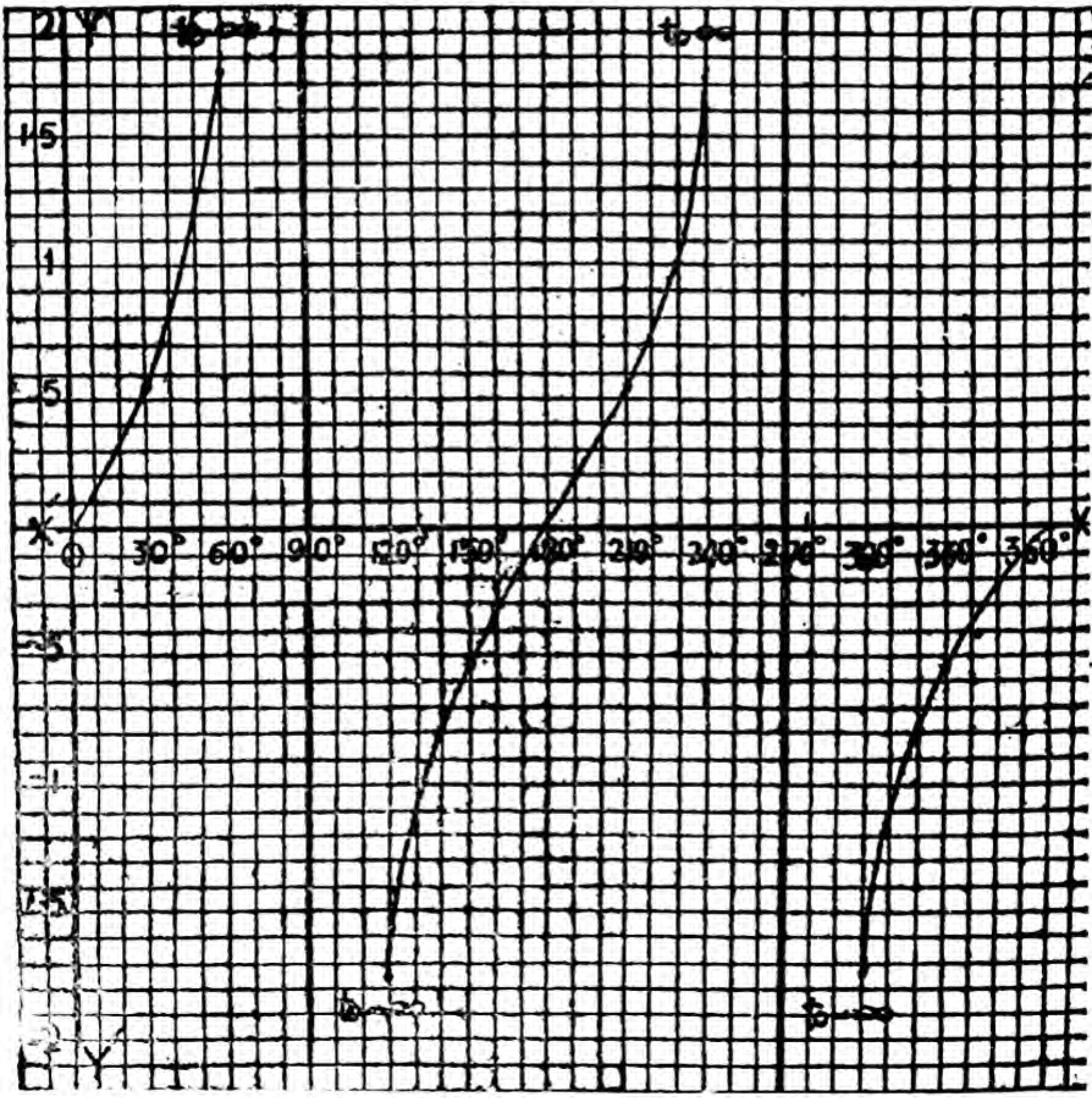
Note. For negative angles the table can be made out with the help of the formula $\cos(-\theta)=\cos\theta$.

9 To draw the graph of tan θ as θ varies from 0° to 360°.

(*i*) Put $y=\tan\theta$. Tabulating values we have :—

θ	0°	30°	60°	$90^\circ-0^\circ$ $90^\circ+0^\circ$	120°	150°	180°	210°	240°	$270^\circ-0^\circ$ $270^\circ+0^\circ$	300°	330°	360°	
y or $\tan\theta$	0	·58	1·7	$+\infty$ $-\infty$	−1·7	−·58	0	·58	1·7	$+\infty$ ∞	−1·7	−·58	0	

(*ii*) Let each small division along OX = 10°
and let ,, ,, ,, ,, OY = ·1
(*iii*) Plot the points given by the above table and draw a smooth curve with a free hand as shown below :—



Cor 1. From the graph of tan θ, find the value of tan 20°.

Cor. 2. From the graph of tan θ, find the angle whose tangent is 1·5.

Note. For negative angles the table can be made out with the help of the formula $\tan(-\theta) = -\tan\theta$.

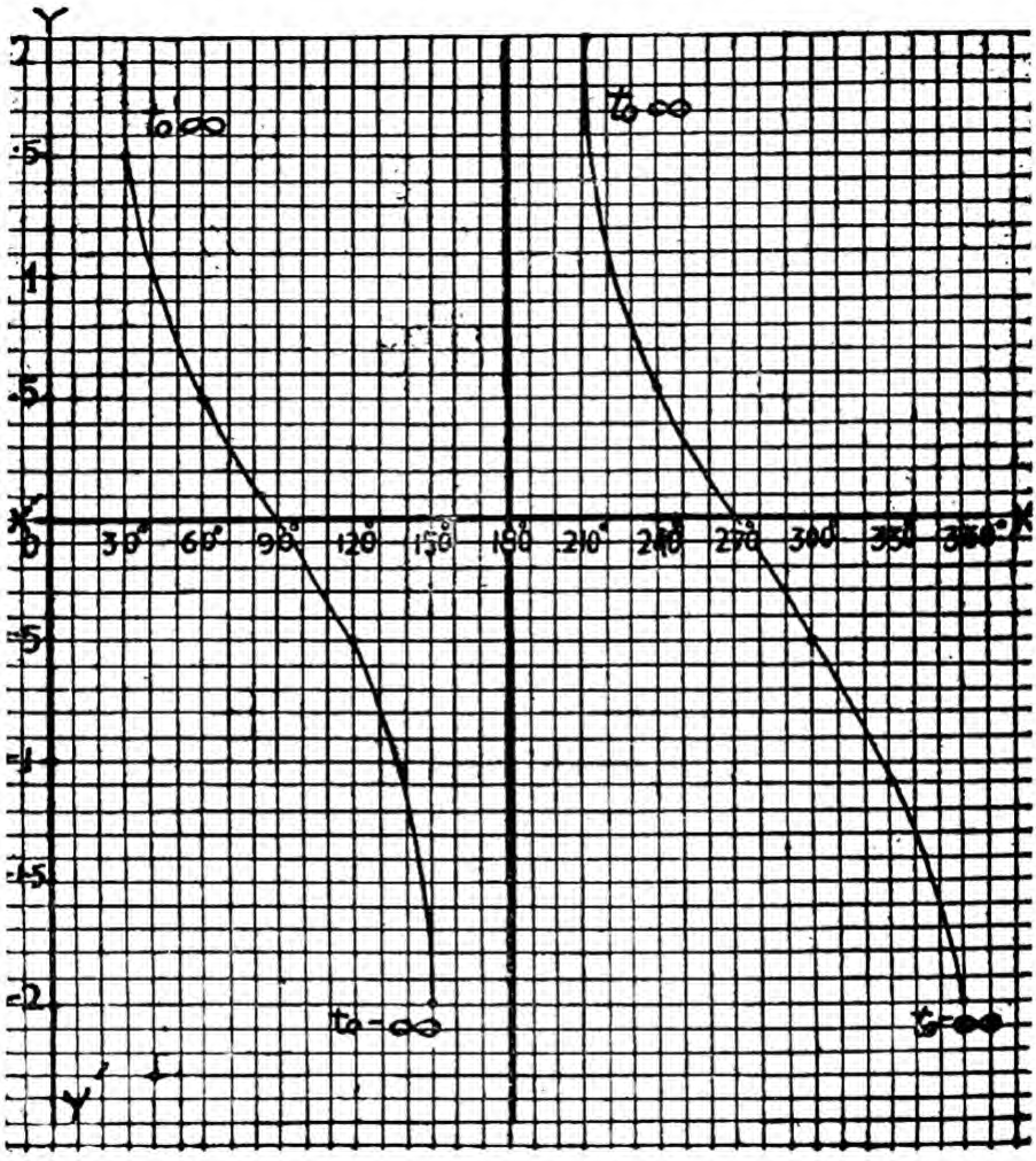
10. To draw the graph of cot θ as θ varies from 0° to 360°.

(*i*) Put $y = \cot\theta$. Tabulating values we have :—

θ	0°	30°	60°	90°	120°	150°	$180^\circ-0^\circ$ $160^\circ+0^\circ$	210°	240°	270°	300°	330°	360°	
y or $\cot\theta$	∞	1·7	·58	0	−·58	−1·7	$-\infty$ $+\infty$	1·7	·58	0	−·58	−1·7	$-\infty$	

(*ii*) Let one small division along OX $= 10^\circ$
and let ,, ,, ,, ,, OY $=$ ·1

(*iii*) Plotting the pts. we get the graph as shown below :–



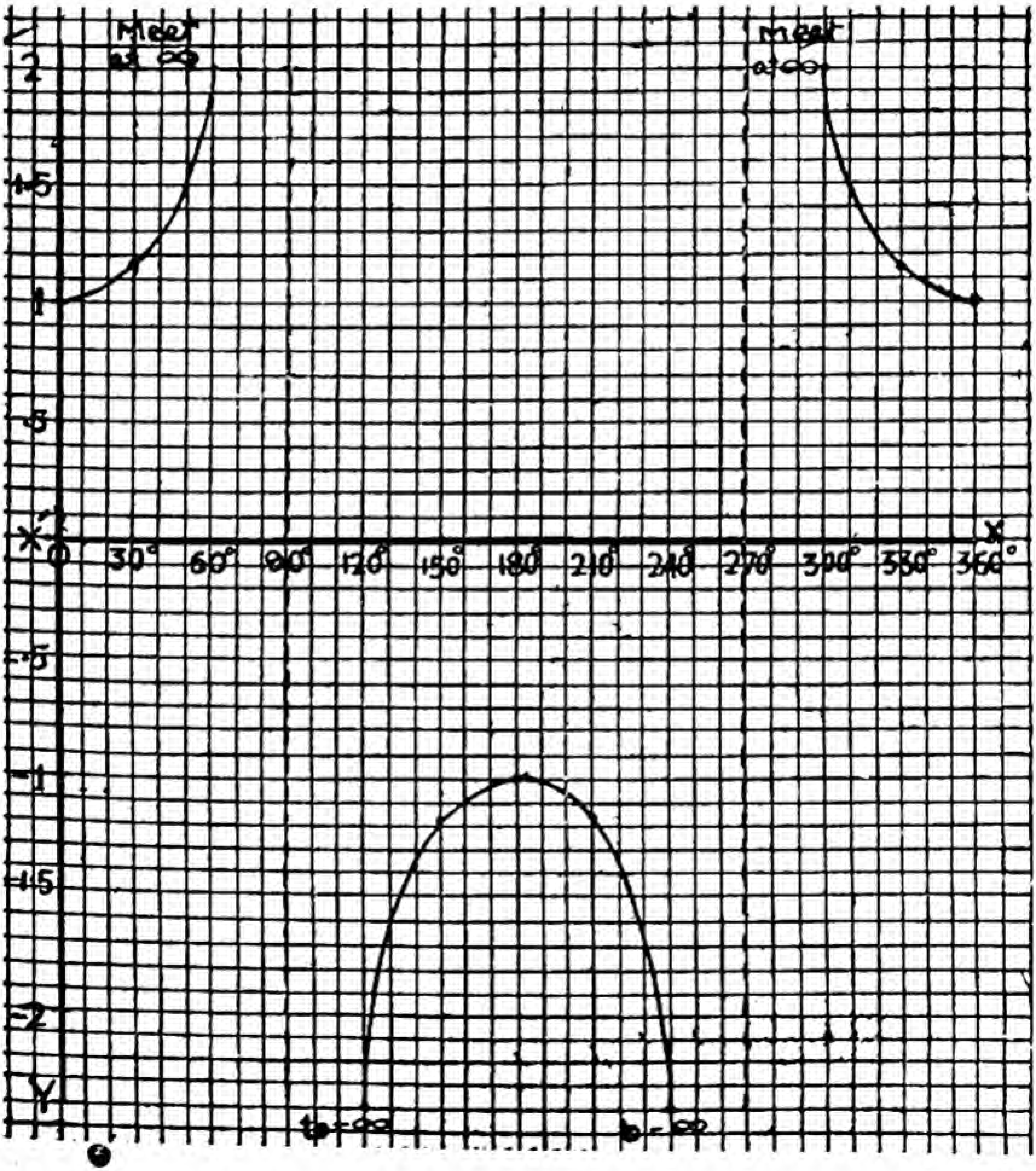
Cor. From the graph of $\cot\theta$, find the angle whose cotangent is 2.

11. To draw the graph of sec θ as θ increases from 0° to 360°.

Put $y=\sec\theta$. Tabulating values we have :

θ	$0°$	$30°$	$60°$	$90°-0°$ $90°+0°$	$120°$	$150°$	$180°$	$210°$	$240°$	$270°-0°$ $270°+0°$	$300°$	$330°$	$360°$
y or sec. θ	1	1·2	2	$+\infty$ $-\infty$	−2	−1·2	−1	−1·2	−2	$-\infty$ $+\infty$	2	1·2	1

Plotting the points we get the graph as shown below :—

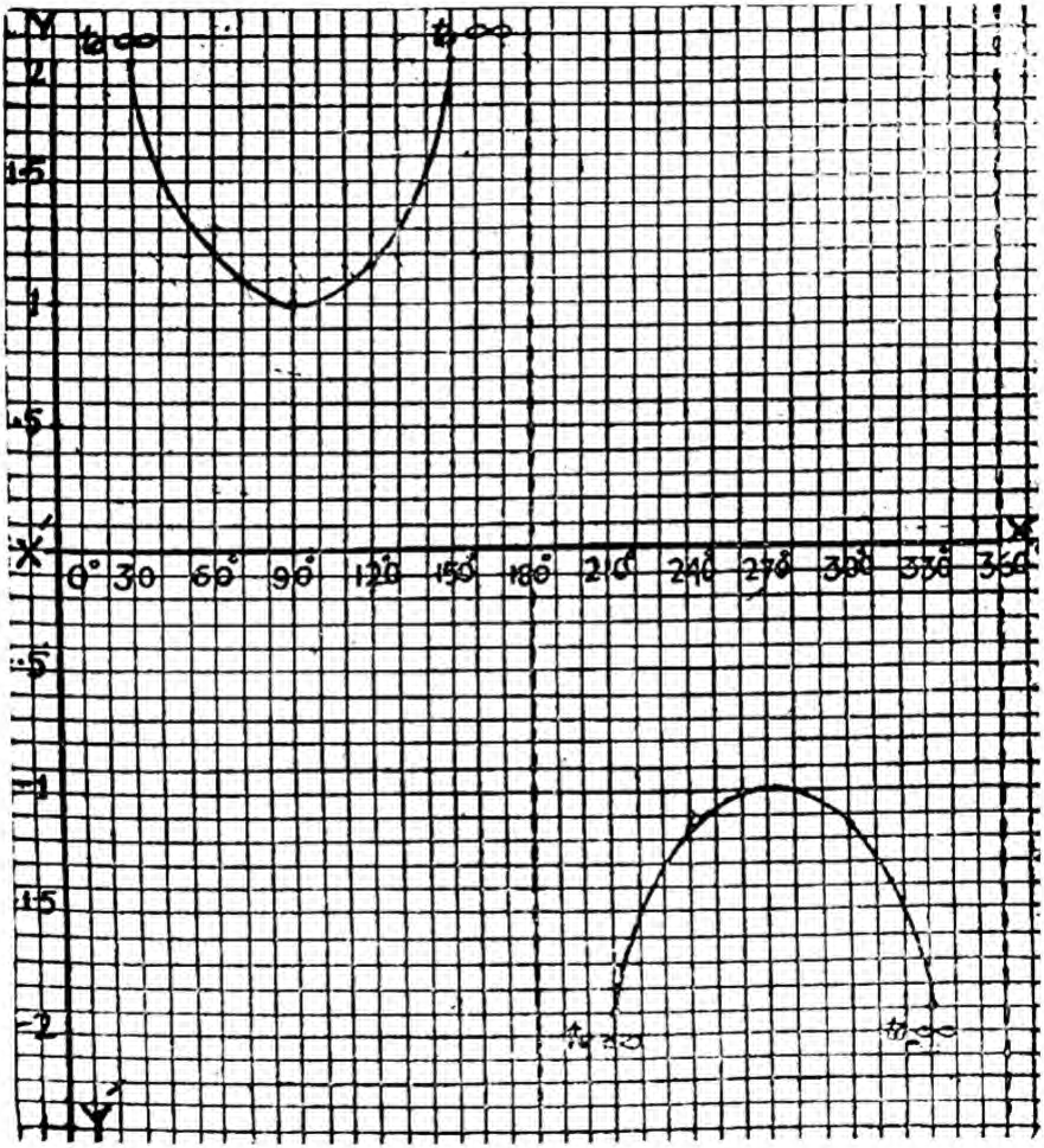


12. To draw the graph of cosec θ as θ increases from 0° to 360°.

Put y=cosec θ. Tabulating values we have :—

θ	0°	30°	60°	90°	120°	150°	$180^\circ-0^\circ$ $180^\circ+0^\circ$	210°	240°	270°	300°	330°	360°	
y or cosec θ	∞	2	1·2	1	1·2	2	$+\infty$ $-\infty$	-2	$-1{\cdot}2$	-1	$-1{\cdot}2$	-2	$-\infty$	

Plotting the points we get the graph as shown below :—



13. To solve an equation graphically.

The following procedure should be adopted.

First Step Put each side of the equation equal to y and draw the graphs of the two equations, thus obtained, on the same scale and with the same axes.

Second Step Read the abscissa of the point of intersection and find its value in degrees according to the scale fixed.

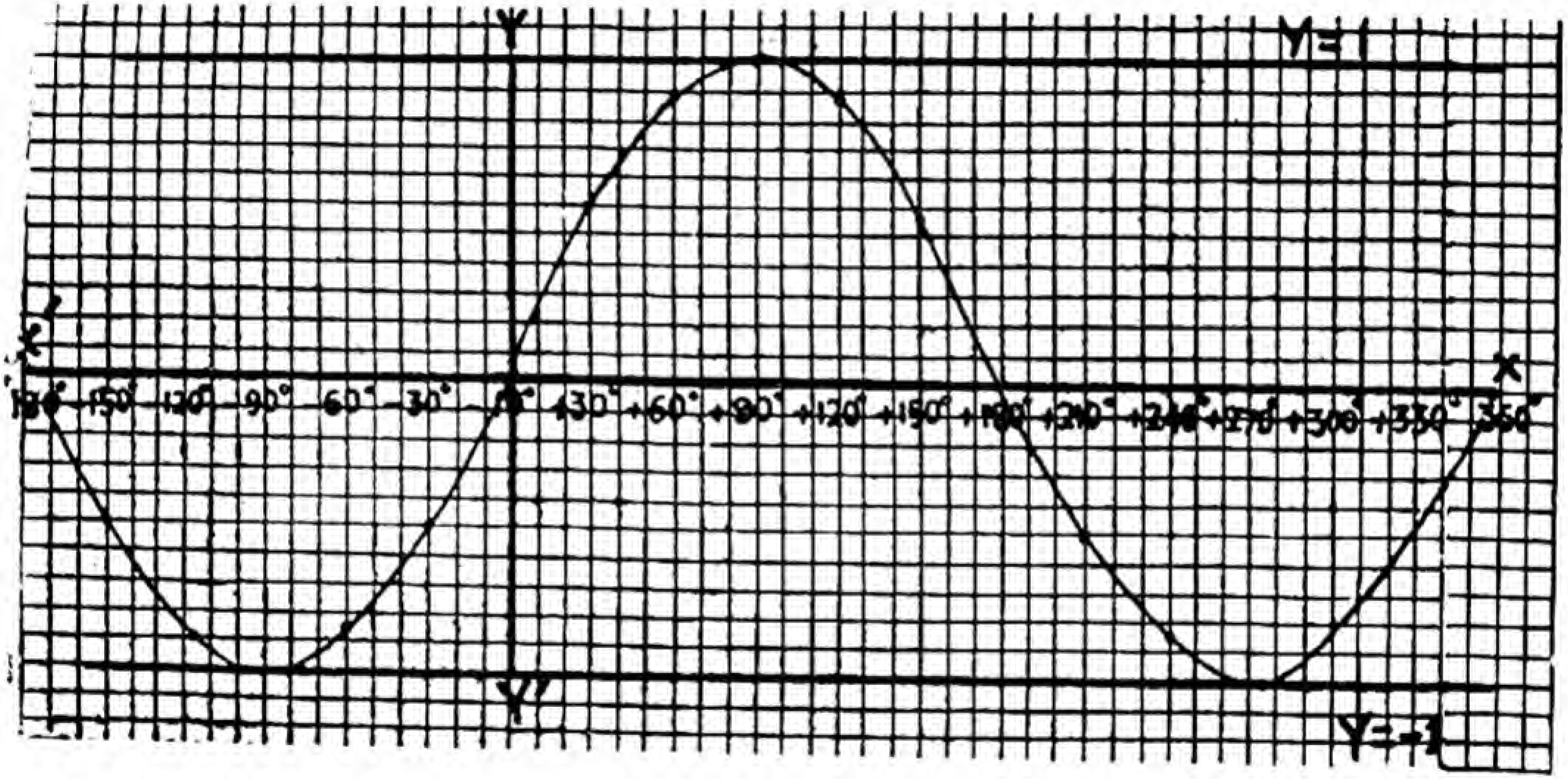
Ex. 1. Draw the graph of $\sin x$ as x varies from $-\pi$ to 2π and solve graphically $\operatorname{cosec}^2 x = 1$.

We have to tabulate values between $-180°$ and $360°$.

Table of values is :—

x	$-180°$	$-150°$	$-120°$	$-90°$	$-60°$	$-30°$	0	$30°$	$60°$	$90°$	$120°$	$150°$	$180°$	$210°$	$240°$	$270°$	$300°$	$330°$	$360°$	
$\sin x$	0	$-{\cdot}5$	$-{\cdot}87$	-1	$-{\cdot}87$	$-{\cdot}5$	0	${\cdot}5$	${\cdot}87$	1	${\cdot}5$	${\cdot}87$	0	$-{\cdot}5$	$-{\cdot}87$	-1	$-{\cdot}87$	$-{\cdot}5$	0	

Plotting the points we have the graph as shown below :—



(*ii*) To solve the equation $\operatorname{cosec}^2 x=1$ is the same as to solve $\sin^2 x=1$ *i. e.* $\sin x=\pm 1$.

(*i*) Take $\sin x=1$. Put $y=1$. Draw the graph of $y=1$ with the same axes and scale as for the graph of $y=\sin x$. It will be a straight line parallel to OM at a unit distance. At the points of intersection of the sine graph with this line draw the perpendiculars to X'OX and read the corresponding values of x in degrees. This will be found$=90°$ *or* $\frac{\pi^c}{2}$.

(*ii*) Now taking $\sin x=-1$, and drawing similarly the graph of $y=-1$, we get a straight line parallel to X'OX at a distance of -1. At the points of intersection we read the values of x to be $\frac{-\pi}{2}$ and $\frac{3\pi}{2}$.

Hence $x=\pm\frac{\pi}{2}$ and $\frac{3\pi}{2}$ are the solutions.

Exercise 6.

1. Trace the variations of $\sin\theta$ as θ varies from $-\pi$ to π and exhibit them by means of a graph. Use the graph to read values of $\sin 35°$ and $\sin 65°$. (P. U. 1942 S.)

2. Draw the graph of $\cos x$ as x lies between $-\pi$ and π and use the graph to solve the equations :—

(*i*) $\cos x=\frac{4}{5}$ (*ii*) $\cos x=\frac{-3}{5}$ (P. U.)

3. Draw the graph of $\sin x$ as x varies from $-180°$ to $180°$ and locate on the graph the values of x for which $\sin^2 x=\frac{1}{2}$. (P. U. 1945)

[**Hint.** $\sin x=\pm\frac{1}{\sqrt{2}}=\pm\cdot 7$; now read x]

4. Trace the variations of $\sin 2x$ as x varies from $0°$ to $180°$ and draw the graph of $y=\sin 2x$.

[Hint. Trace changes between the intervals $0°$ to $45°$, $45°$ to $90°$,......

For graph tabulate values for $x=0°$, $15°$, $30°$, $45°$.......

5. Draw in the same figure the graphs of $y=\sin x$ and $y=2\cos x$ between $0°$ and $180°$. Read the values of x where the graphs intersect. (D. U.)

6. (*a*) Trace the changes in the sign and magnitude of $\tan A$ as A increases from $0°$ to $360°$. (J. & K. U. 1949)

(*b*) Draw the graph of $\tan \theta$ as θ varies from 0 to 2π. Show from the graph the period of $\tan \theta$.

7. Draw the graph of $y=\tan x$ between 0 and 2π and locate on the graph the values of x for which,

(*i*) $3\tan^2 x=1$. (*ii*) $\tan x=\cot x$. (J &K. U. 1957)

[Hint. (*ii*) $\tan x=\cot x$ gives $\tan^2 x=1$ *i. e.* $\tan x=\pm 1$].

8. Draw the graph of $\sin x+\cos x$ from $0°$ to $180°$. (J, & K. U. 1950)

9. Draw the graph of $\sec \theta$ as θ varies from 0 to 2π. Illustrate the formula $\sec(\pi-\theta)=-\sec\theta$ from your graph. (J. & K. U. 1953)

10. Solve graphically the equation $\tan x=x$ between $x=0$ and $x=\frac{\pi}{2}$. (Calcutta U. 1945)

11. Trace the changes in the values of $\operatorname{cosec}\theta$ as θ changes from $0°$ to $180°$ and illustrate them by means of a graph. (P U. 1951)

12. Determine graphically the roots of the equation $\tan x=2-\frac{4}{\pi}x$ which lie between 0 and π, x being in radians.

(P. U. 1934)

[**Hint.** Draw graphs of $y = \tan x$ and $y = 2 - \frac{4}{\pi} x$ and tabulate values for $x = 0, \frac{\pi}{6}, \frac{\pi}{3}, \frac{\pi}{2}, \frac{2\pi}{3}, \frac{5\pi}{6}, \pi$.]

13. From the graph of tan x and sin x deduce that for

$0 < x < \frac{\pi}{2}$, $\sin x < x < \tan x$. (P. U 1933)

14. Solve graphically $\cos x = x$, when x is measured in radians and lies between 0 and 2π. (P. U. 1938)

15. Trace the changes in the values of $\sin\theta + \sqrt{3}\cos\theta$ as θ changes from 0 to x.

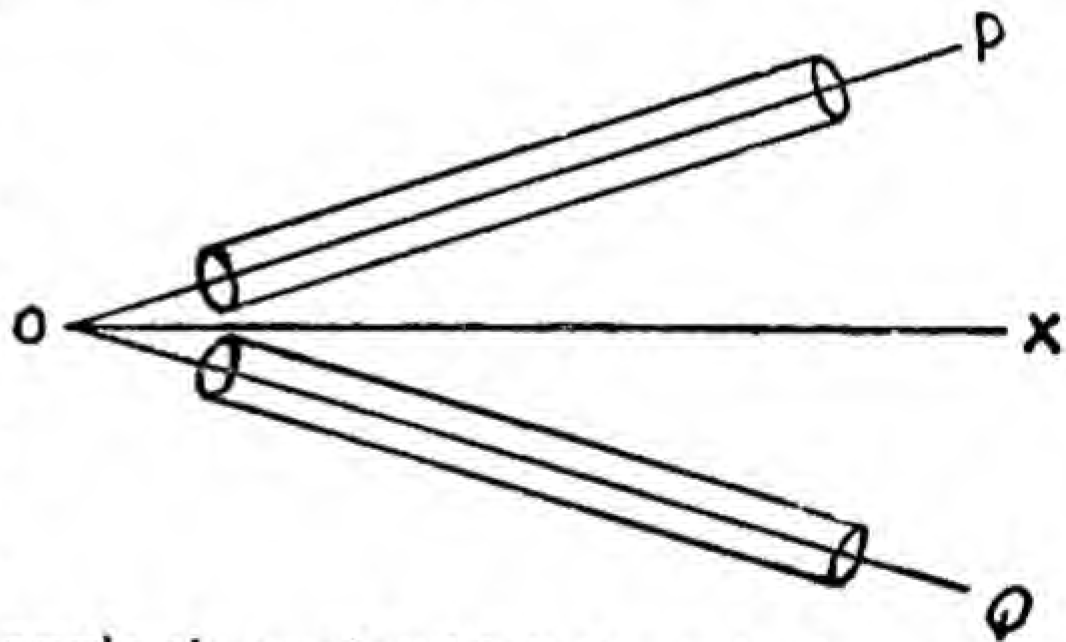
CHAHTER V

Simple problems in heights and distances.

1. Trigonometry is used in finding the distances and heights of inaccessible objects like the sun, the moon, the planets, mountain tops, towers or trees by measuring convenient lines and angles. Such problems occur in land surveying. The angles are measured by instruments such as Theodolite and Sextant.

Here we shall consider only easy problems requiring the use of T-ratios of 30°, 45° and 60°.

2. Definition :— Let P be an object on a higher level to be observed from O and let OX be the horizontal line through O. To observe the object P from O through a tube, it has to be raised through an $\angle XOP$. This angle through which the tube is elevated above the horizontal line is called the **angle of elevation** or simply the *elevation* of the object P.



On the other hand, if an object Q, on a lower level, is to be observed through the tube, it has to be depressed through an $\angle XOQ$, below the horizontal. This angle, through which the tube is depressed below the horizontal line, is called the **angle of depression**.

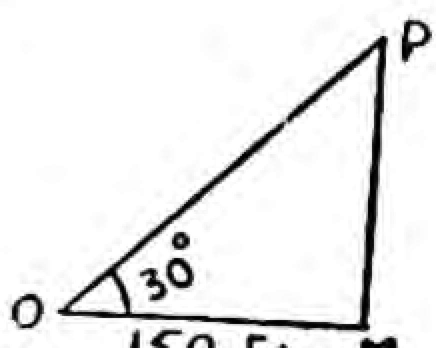
Ex. 1. The elevation of the top of a tree from a point on the ground 150 ft. from it is 30°. Find the height of the tree.

Let O be the observer and MP the tree so that OM=150 ft. and $\angle MOP = 30°$.

From the rt. $\angle$d $\triangle$ OMP,

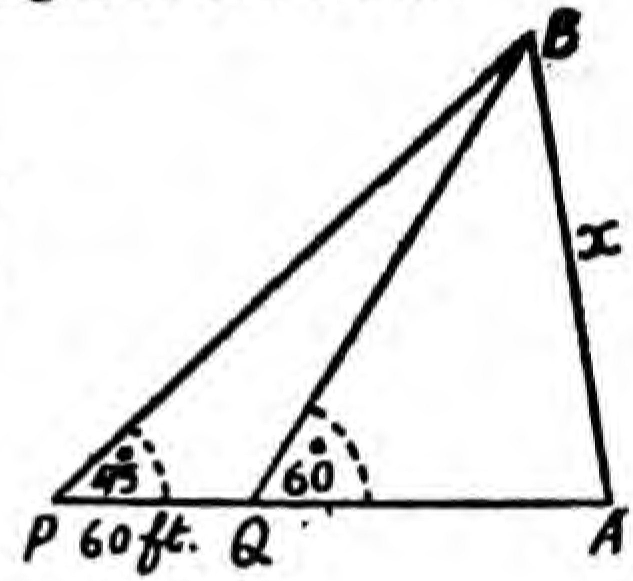
$$\frac{MP}{OM} = \tan 30°$$

or, $MP = OM \tan 30° = 150 \times \frac{1}{\sqrt{3}}$



$$=\frac{150\angle 3}{3}=50\angle 3$$
$$=50\times 1{\cdot}732=86\ 6 \text{ ft.}$$

Ex. 2. From a point on the level ground the elevation of the top of a tower is 45° and on coming 60 ft. nearer the tower, the elevation is 60°. Find the height of the tower.



Let P and Q be the two places of observation and let AB be the height of the tower. Let AB$=x$ ft.

then $\tan 45°=\dfrac{x}{\text{PA}}=1$

$\therefore$ PA$=x$

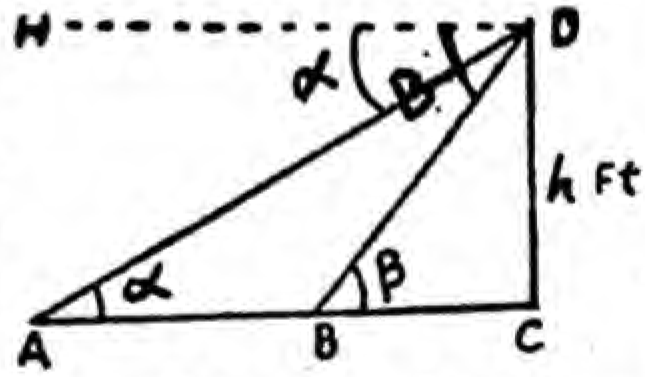
and $\tan 60°=\dfrac{\text{AB}}{\text{AQ}}\ \therefore \dfrac{x}{x-60}=\sqrt{3}$

which gives $x=\dfrac{60\sqrt{3}}{\sqrt{3}-1}=\dfrac{60\sqrt{3}\ (\sqrt{3}+1)}{2}$

$=30\ (3+\sqrt{3})=141{\cdot}96$ ft.

Ex. 3. From the top of a cliff h ft. high the angles of depression of two ships at sea in a line with the foot of the cliff are α and β respectively. Show that the distance between the ships is h ($\cot\alpha-\cot\beta$

(J. & K. U. 1949)



Let CD be the cliff and A and B the positions of the ships. Let DH be the horizontal line through D, then

$\angle$HDA$=\alpha=$alt. $\angle$DAC

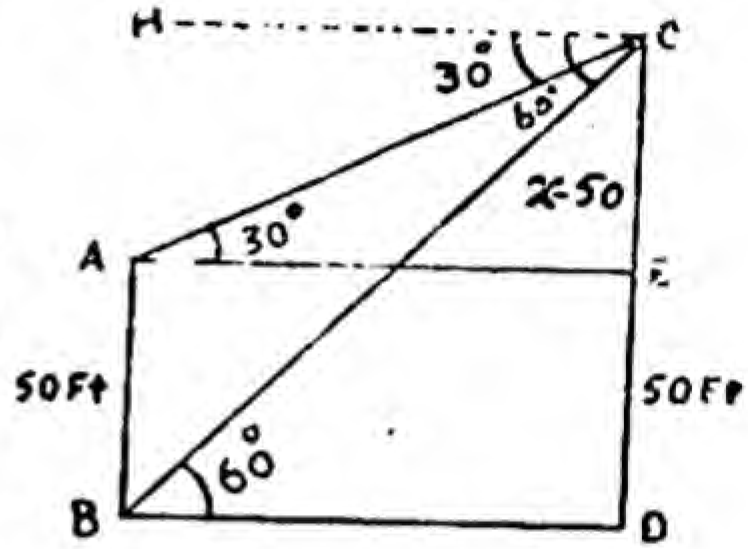
and $\angle$HDB$=\beta=$alt. $\angle$DBC

$\therefore$ From $\triangle$ ACD, $\dfrac{\text{AC}}{\text{DC}}=\cot\alpha$ *i. e.* AC$=h\cot\alpha$

and from $\triangle$BCD, $\dfrac{\text{BC}}{\text{CD}}=\cot\beta$ *i. e.* BC$=h\cot\beta$

$\therefore$ AB=AC−BC=h ($\cot\alpha-\cot\beta$).

Ex. 4. From the top of a tower, the angles of depression of the top and the bottom of a building 50 ft. high are 30° and 60° respectively. Find the height of the tower and its distance from the building.



Let AB be the building and CD the tower.

Draw AE$\perp$CD.

Let CD=x ft. and BD=y ft.

From $\triangle$ACE, $\tan 30° = \frac{CE}{AE}$

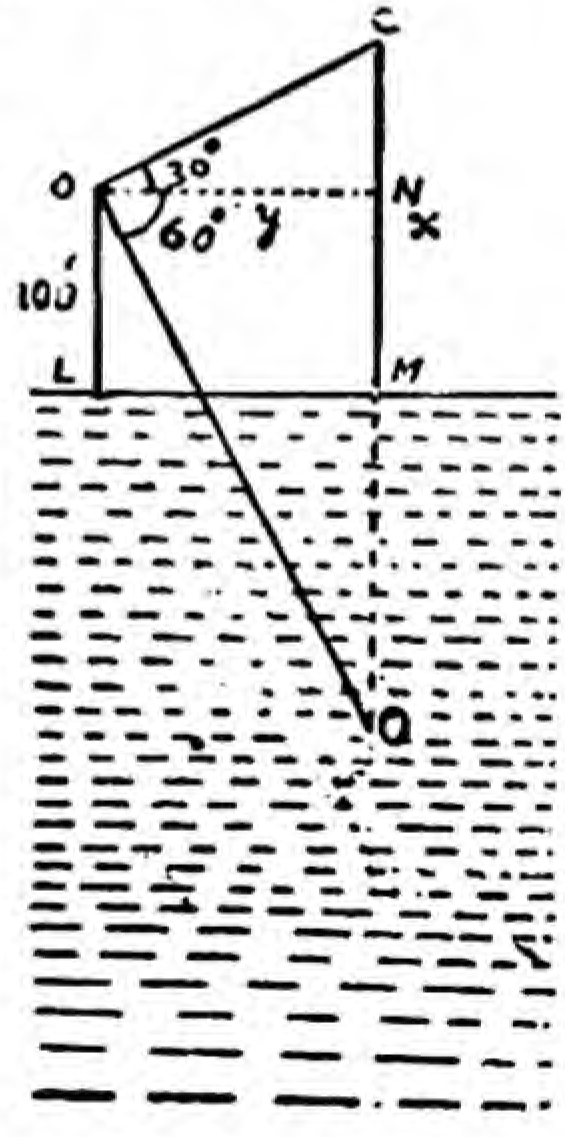
i.e. $\frac{x-50}{y} = \frac{1}{\sqrt{3}}$ ......(1)

From $\triangle$ BCD, $\tan 60° = \frac{x}{y}$

i. e. $x = y\sqrt{2}$ ......(2)

Solving (1) and (2), x=75 ft. and y=43·3 ft.

Ex 5. The angle of elevation of a cloud from a point 100 ft. above a lake is 30° and the angle of depression of its reflection in the lake is 60°. Find the height of the cloud.



Let O be the observer, and C the cloud. Draw OL and CM$\perp$s to LM, the surface of the lake, then

OL=100 ft.

Draw ON $\perp$ MC, so that $\angle$NOC=30°.

Produce CM to D, such that MD=MC,

then D is the reflection of the cloud (By the law of reflection).

and NOD=60°

Let MC=x, and ON=y

Now CN=CM−NM=CM−OL
$=x-100$

and DN=DM+MN=x+100

From $\triangle$ONC, $\tan 30° = \frac{NC}{ON}$
$= \frac{x-100}{y}$ ...(1)

From $\triangle$OND, $\tan 60° = \dfrac{ND}{ON} = \dfrac{x+100}{y}$ ......(2)

$\therefore$ Dividing (1) by (2), $\dfrac{x-100}{x+100} = \dfrac{\tan 30°}{\tan 60°} = \dfrac{1}{\sqrt{3} \times \sqrt{3}}$
$$= \frac{1}{3}$$

which gives $x = 200$ ft.

Exercise 7.

1. The elevation of the top of a tree from a point on the ground 50 ft. from it is 45°. Find the height of the tree.

2. The angle of elevation of a kite above the ground is 60° and the length of the string is 120 yds. Find the height of the kite above the ground.

3. The angle of elevation of the top of a flag staff, on a fort, from a point 200 ft. from the foot of the fort is 30°, find the height of the flagstaff.

4. From the top of a tower 100 ft. high the angles of depression of the top and the bottom of a house are 30° and 45°. Find the height of the house and its distance from the foot of the hill.

5. From the top of a building 20 ft. high the elevation of the top of a tower is 60° and the depression of its foot is 30°; find the height and the distance of the tower.

6. A man, whose eye is 6 ft. above the level of a lake, sees the top of a tree at an elevation af 30° and its image in water at a depression of 45°. Find the height of the tree above the water level.

7. The altitude of the top of a chimney is 30°; approaching 200 ft. towards it, its altitude becomes 45°. Find the height of the chimney. (J. & K U. 1951)

8. The angles of depression of two motor cars standing on a road and observed from the top of a tower are 45° and 60° respectively. If the cars and the tower are in a vertical plane, and the cars 300 ft. apart, find the height of the tower. (J, & K. U. 1952)

9. The upper part of a tree broken over by wind makes 30° with the ground and the top touches the ground at a distance of 20 ft. from the foot of the unbroken part. Find the original height of the tree.

10. A telegraph post 50 ft. high is observed from two points in horizontal line with the foot of the post and on opposite sides of it. The tangents of the angles of elevation at these points are $\frac{3}{4}$ and $\frac{4}{3}$. Find the distance between the points.

11. From a light-house the angles of depression of two ships on opposite sides of the light house are observed to be 30° and 45°. If the height of the light-house be 300 ft., find the distance between the ships if the line joining them passes through the foot of the light house. (P. U. 1941)

12. A person standing on the bank of a river observes that the angle subtended by a tree on the opposite bank is 60°; when he retires 40 ft. from the bank he finds the angle to be 30°. Find the height of the tree and breadth of the river. (P. U. 1942 S.)

13. The height of a house subtends right angle at an opposite window, the top being 60° above the horizontal straight line. Find the height of the house, the street being 40 ft. wide.

14. The angle of elevation of a tower from a point A due south of it is x and from a point B due east of A is y. If $AB = l$, show that the height of the tower is given by $h^2 (\cot^2 y - \cot^2 x) = l^2$ (P. U. 1943)

15. What is the angle of elevation of the sun when the length of the shadow of a pole is $\sqrt{3}$ times the height of the pole. (A. U. 1946)

16. If the angle of elevation of a cloud from a point h ft. above the lake be β, and the angle of depression of its reflection in the lake α, prove that the height of the cloud is $\dfrac{h \sin (\alpha + \beta)}{\sin (\alpha - \beta)}$. (D.U.)

17. The angles of depression of two boats in the Dal lake observed in the same direction from the top of Shankaracharya hill are 30° and 60° respectively. If the

top of the hill be 1200 ft. vertically above the level of the lake, find the distance between the boats and the distance of the second boat from the observer. (J & K U. 1952 S)

18. The angles of elevation of the top of a tower observed by two observers standing on a road, on the opposite sides of the tower, are 30° and 60° respectively. If the observers and tower are in the same vertical plane, and the observers are 500 ft. apart, find the height of the tower.

(J. & K. U. 1953)

19. From the top of a tower 100 ft high, the angles of depression of two objects due north of the tower are 60° and 45°. Find the distance between the objects to the nearest foot. (J. & K. U. 1954)

20. From the top of a cliff, 200 ft. high, the angles of depression of the top and bottom of a tower are observed to be 30° and 60° respectively, find the height of the tower.

(J. & K. U. 1960)

21. A person standing on the bank of a river finds that the angle of elevation of the top of a cliff on the opposite bank is 60°. On going back 100 yds, he finds that the angle of elevation is only 30°. Find the height of the cliff and breadth of the river. (J. & K. U. 1956)

22. Two posts of the same height stand on either side of a road 120 ft. wide. At a point in the road between the posts, the elevations of the tops of posts of the pillars are 60° and 30°. Find the height of the posts and the position of the point.

(J. & K. U. 1957)

23. The shadow of a tower standing on a level plane is found to be 60 feet longer, when the sun's altitude is 30°, than when it is 45°. Prove that the height of the tower is $30\,(1+\sqrt{3})$

(J. & K. U. 1961)

CHAPTER VI.

Trigonometric ratios of sum or difference of angles.

or

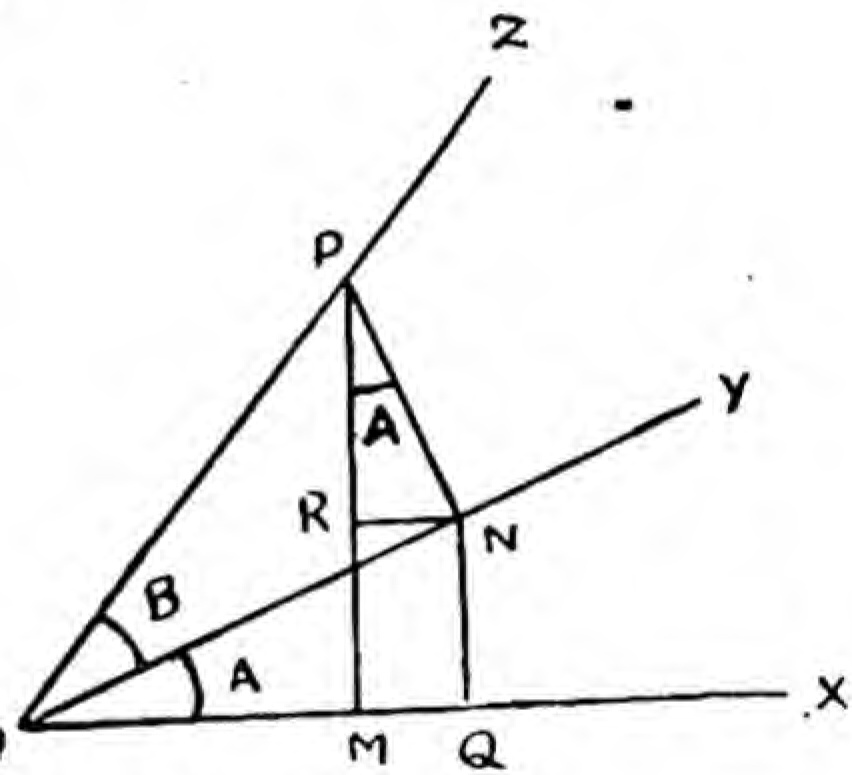
Addition and Subtraction theorems.

To prove geometrically that

$\sin(A+B)=\sin A\cos B+\cos A\sin B$;

$\cos(A+B)=\cos A\cos B-\sin A\sin B$;

$\tan(A+B)=\dfrac{\tan A+\tan B}{1-\tan A\tan B}$.



Let a revolving line, starting from OX, trace out $\angle XOY=A$ and then revolve further through $\angle YOZ=B$, so that $\angle XOZ=A+B$.

From any point P in the final position OZ of the revolving line, draw PM and PN $\perp$s to OX and OY ; from N draw NQ and NR $\perp$s OX and MP respectively.

Then $\angle RPN=90^\circ-\angle PNR$ $=\angle RNO=$ alt. $\angle NOQ=\angle A.$

$$\therefore \sin(A+B)=\sin\angle XOZ=\frac{MP}{OP}=\frac{MR+RP}{OP}$$

$$=\frac{QN+RP}{OP}\quad (\because MR=QN)$$

$$=\frac{QN}{OP}+\frac{RP}{OP}=\frac{QN}{ON}\cdot\frac{ON}{OP}+\frac{RP}{NP}\cdot\frac{NP}{OP}$$

$=\sin A\cos B+\cos\angle RPN\sin B$

$=\sin A\cos B+\cos A\sin B.$

Again, $\cos(A+B)=\cos\angle XOZ=\dfrac{OM}{OP}=\dfrac{OQ-MQ}{OP}$

$$=\frac{OQ-RN}{OP}=\frac{OQ}{OP}-\frac{RN}{OP}\quad(\because MQ=RN)$$

$$=\frac{OP}{ON}\cdot\frac{ON}{OP}-\frac{RN}{NP}\cdot\frac{NP}{OP}$$

$$=\cos A\cos B-\sin\angle RPN\sin B$$

$$=\cos A\cos B-\sin A\sin B.$$

$$\tan(A+B)=\tan\angle XOZ=\frac{MP}{OM}=\frac{MR+RP}{OQ-MQ}=\frac{QN+RP}{OQ-RN}$$

$$=\frac{\frac{QN}{OQ}+\frac{RP}{OQ}}{1-\frac{RN}{OQ}}\text{(Dividing Numr. and Denr. by OP)}$$

$$=\frac{\tan A+\frac{RP}{OQ}}{1-\frac{RN}{RP}\cdot\frac{RP}{OQ}}=\frac{\tan A+\frac{RP}{OQ}}{1-\tan A.\frac{RP}{OQ}}$$

But from similar $\triangle$s RPN and QON, $\frac{RP}{OQ}=\frac{NP}{ON}=\tan B$

$$\therefore\ \tan(A+B)=\frac{\tan A+\tan B}{1-\tan A\tan B}.$$

or thus : $\tan(A+B)=\frac{\sin(A+B)}{\cos(A+B)}$

$$=\frac{\sin A\cos B+\cos A\sin B}{\cos A\cos B-\sin A\sin B}$$

$$=\frac{\frac{\sin A}{\cos A}+\frac{\sin B}{\cos B}}{1-\frac{\sin A}{\cos A}.\frac{\sin B}{\cos B}}$$

(Dividing Numr. and Denr. by cos A cos B)

$$=\frac{\tan A+\tan B}{1-\tan A\ \tan B}$$

Cor. 1. $\tan(45^\circ+A)=\frac{1+\tan A}{1-\tan A}$

Cor. 2: $\cot(A+B)=\dfrac{\cot A.\cot B-1}{\cot B+\cot A}$

[Hint. $\cot(A+B)=\dfrac{\cos(A+B)}{\sin(A+B)}$, expand and divide [Nnmr. and Denr, by sin A sin B]

Note. 1. In the above proof angles A, B, A+B are all acute. The construction and proof are the same word by word for angles of greater magnitudes, due attention being paid to the signs of lengths involved.

It will be a good exercise for the student to draw fig. for the cases when (*i*) A, B are acute but A+B is obtuse and (*ii*) when A and B are both $>90°$.

Note. 2. Remember that *sin, cos, tan* are not multipliers and hence it is wrong to say: $\sin(A+B)=\sin A+\sin B$.

We can prove the above results in the following way also.

Case I. When A, B, A+B are acute angles. the above proof of art. 1 gives the results.

Case II. When one of the two component angles, say A, is obtuse,

i.e., $A=A_1+90°$ where A_1 is acute. So that $\sin A=\cos A_1$.

$$\begin{aligned}\therefore \sin(A+B)&=\sin(A_1+90°+B)=\cos(A_1+B)\\&=\cos A_1\cos B-\sin A_1\sin B\\&=\cos(A-90°)\cos B-\sin(A-90°)\sin B\\&=\sin A\cos B+\cos A\sin B\end{aligned}$$

$$\begin{aligned}\text{Also } \cos(A+B)&=\cos(A_1+90°+B)=-\sin(A_1+B)\\&=-\sin A_1\cos B-\cos A_1\sin B\\&=-\sin(A-90°)\cos B-\cos(A-90°)\sin B\\&=\cos A\cos B-\sin A\sin B.\end{aligned}$$

Similarly, if B is obtuse we can prove the above formulae by putting $B=B_1+90°$.

In the same way we can prove the formulae when A and B both are obtuse.

Thus the formulae of art. 1 are true when angles A and B lie between 0° and 180°.

Case III. If the angles A and B lie between 0° and 270° we should put $A=90°+A_2$ or $B=90°+B_2$, where A_2 and B_2 lie between 0° and 180° and then the results follow in a manner similar to that of case II.

Proceeding in this way we can show that the theorems are true for all values of A and B.

Ex 1. Find sin 75°, cos 75° and tan 75°.

$$\sin 75°=\sin(45°+30°)=\sin 45° \cos 30°+\cos 45° \sin 30°$$

$$=\frac{1}{\sqrt{2}}\cdot\frac{\sqrt{3}}{2}+\frac{1}{\sqrt{2}}\cdot\frac{1}{2}=\frac{\sqrt{3}+1}{2\sqrt{2}}.$$

$$\cos 75°=\cos(45°+30°)=\cos 45° \cos 30°-\sin 45° \sin 30°$$

$$=\frac{1}{\sqrt{2}}\cdot\frac{\sqrt{3}}{2}-\frac{1}{\sqrt{2}}\cdot\frac{1}{2}=\frac{\sqrt{3}-1}{2\sqrt{2}}.$$

$$\tan 75°=\tan(45°+30°)=\frac{\tan 45°+\tan 30°}{1-\tan 45° \tan 30°}$$

$$=\frac{1+\frac{1}{\sqrt{3}}}{1-\frac{1}{\sqrt{3}}}=\frac{\sqrt{3}+1}{\sqrt{3}-1}=\frac{(\sqrt{3}+1)(\sqrt{3}+1)}{(\sqrt{3}-1)(\sqrt{3}+1)}$$

$$=\frac{3+1+2\sqrt{3}}{3-1}=2+\sqrt{3}.$$

Ex. 2. Show that $\dfrac{\sin(\alpha+\beta)}{\cos\alpha\cos\beta}=\tan\alpha+\tan\beta.$

$$\frac{\sin(\alpha+\beta)}{\cos\alpha\cos\beta}=\frac{\sin\alpha\cos\beta+\cos\alpha\sin\beta}{\cos\alpha\cos\beta}$$

$$=\frac{\sin\alpha\cos\beta}{\cos\alpha\cos\beta}+\frac{\cos\alpha\sin\beta}{\cos\alpha\cos\beta}$$

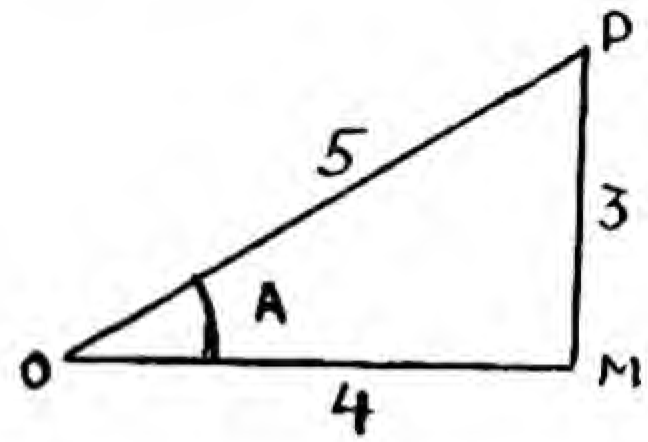
$$=\tan\alpha+\tan\beta.$$

Ex. 3. Given $\sin A=\frac{3}{5}$ and $\sin B=\frac{5}{13}$, both A and B being acute.

Find (*i*) $\sin (A+B)$,

(*ii*) $\cos (A+B)$,

(*iii*) $\tan (A+B)$.

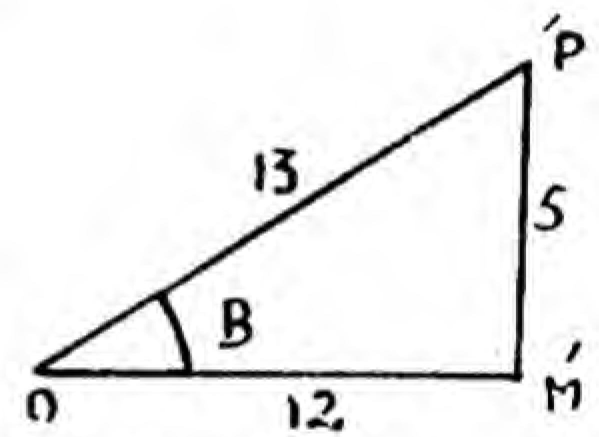


Since A and B are acute, all their T-ratios will be positive.

From Δs OMP and OM′P′,

$$\cos A=\frac{4}{5} \text{ and } \cos B=\frac{12}{13}$$

$$\tan A=\frac{3}{4} \text{ and } \tan B=\frac{5}{12}.$$



(*i*) $\sin (A+B)$

$=\sin A \cos B+\cos A \sin B$

$=\frac{3}{5}\cdot\frac{12}{13}+\frac{4}{5}\cdot\frac{5}{13}=\frac{56}{65}.$

(*ii*) $\cos (A+B)$

$=\cos A \cos B-\sin A \sin B$

$=\frac{4}{5}\cdot\frac{12}{13}-\frac{3}{5}\cdot\frac{5}{13}=\frac{33}{65}.$

(*iii*) $\tan (A+B)=\dfrac{\tan A+\tan B}{1-\tan A \tan B}=\dfrac{\frac{3}{4}+\frac{5}{12}}{1-\frac{3}{4}\cdot\frac{5}{12}}$

$$=\frac{\frac{9+5}{12}}{\frac{48-15}{48}}=\frac{14}{12}\times\frac{48}{33}=\frac{56}{33}.$$

Ex. 4. Prove that (*i*) $\sin (90°+\theta)=\cos \theta$

(*ii*) $\cos (180°+\theta)=-\cos \theta$.

(*i*) $\sin (90°+\theta)=\sin 90° \cos \theta+\cos 90° \sin \theta$

$=1.\cos \theta+(0)\times\sin \theta=\cos \theta.$

(*ii*) $\cos (180°+\theta)=\cos 180° \cos \theta-\sin 180° \sin \theta.$

$=(-1) \cos \theta-(0) \sin \theta.$

$=-\cos \theta.$

Ex. 5. If $A+B=\frac{\pi}{4}$, prove that $(1+\tan A)\ (1+\tan B)=2$

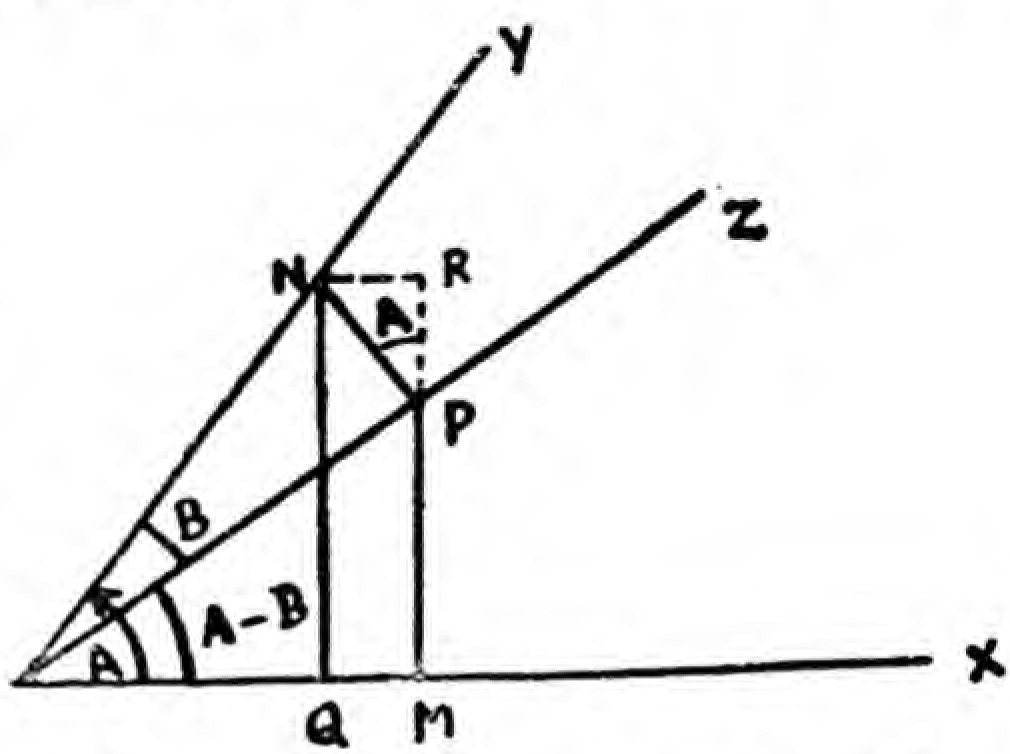
$$\because\ A+B=\frac{\pi}{4}$$

$$\therefore\ \tan(A+B)=\tan\frac{\pi}{4}=1$$

$$or,\quad \frac{\tan A+\tan B}{1-\tan A\tan B}=1$$

or, $\tan A+\tan B=1-\tan A\tan B.$
or, $1+\tan A+\tan B+\tan A\tan B=2.$
or, $(1+\tan A).(1+\tan B)=2.$

2. To prove geometrically that
$$\sin(A-B)=\sin A\cos B-\cos A\sin B$$
$$\cos(A-B)=\cos A\cos B+\sin A\sin B$$
$$\tan(A-B)=\frac{\tan A-\tan B}{1+\tan A\ \tan B}$$ (J. & K. U. 1952)



Let a revolving line, starting from OX, trace out $\angle XOY=A$ and then revolve back through $\angle YOZ=B$, so that $\angle XOZ=A-B$.

From any point P in the final position OZ of the revolving line, draw PM and PN $\perp$s OX and OY. From N draw NQ and NR $\perp$s OX and MP respectively. Then $\angle RPN=90^\circ-\angle PNR=\angle RNY=$ corresp. $\angle NOQ=A$.

Hence $\sin(A-B)=\sin\angle XOZ=\frac{MP}{OP}=\frac{MR-PR}{OP}=\frac{QN-PR}{OP}$

$$=\frac{QN}{OP}-\frac{PR}{OP}=\frac{QN}{ON}.\frac{ON}{OP}-\frac{PR}{NP}.\frac{NP}{OP}$$

$=\sin A \cos B - \cos \angle RPN \sin B$
$=\sin A \cos B - \cos A \sin B.$

Again, $\cos (A-B) = \cos \angle XOP = \dfrac{OM}{OP} = \dfrac{OQ+QM}{OP} = \dfrac{OQ+NR}{OP}$

$$= \frac{OQ}{OP} + \frac{NR}{OP} = \frac{OQ}{ON} \cdot \frac{ON}{OP} + \frac{NR}{NP} \cdot \frac{NP}{OP}$$

$=\cos A \cos B + \sin \angle RPN \sin B$
$=\cos A \cos B + \sin A \sin B.$

$$\tan (A-B) = \frac{MP}{OM} = \frac{MR-PR}{OQ+QM} = \frac{QN-PR}{OQ+NR}$$

$$= \frac{\dfrac{QN}{OQ} - \dfrac{PR}{OQ}}{1 + \dfrac{NR}{OQ}} = \frac{\tan A - \dfrac{PR}{OQ}}{1 + \dfrac{NR}{PR} \cdot \dfrac{PR}{OQ}}$$

$$= \frac{\tan A - \dfrac{PR}{OQ}}{1 + \tan A . \dfrac{PR}{OQ}}$$

But from similar $\triangle$s OQN and RPN, $\dfrac{PR}{OQ} = \dfrac{PN}{ON} = \tan B$

$$\therefore \tan (A-B) = \frac{\tan A - \tan B}{1 + \tan A \tan B}.$$

Or thus : $\tan (A-B) = \dfrac{\sin (A-B)}{\cos (A-B)}$

$$= \frac{\sin A \cos B - \cos A \sin B}{\cos A \cos B + \sin A \sin B}$$

$$= \frac{\dfrac{\sin A}{\cos A} - \dfrac{\sin B}{\cos B}}{1 + \dfrac{\sin A}{\cos A} \cdot \dfrac{\sin B}{\cos B}}$$

(Dividing both the Numr. and Denr. by $\cos A \cos B$)

$$= \frac{\tan A - \tan B}{1 + \tan A \tan B}.$$

Cor. 1. $\tan(45^\circ - A) = \dfrac{1-\tan A}{1+\tan A}$.

Cor. 2. $\cot(A-B) = \dfrac{\cot A \cot B + 1}{\cot B - \cot A}$

Note : In the above proof angles A,B,A−B are all acute, The construction and proof are the same, word by word, for angles of any magnitude, due attention being paid to the signs of lengths involved.

3. **To prove that :—**

(*i*) $\sin(A+B)\sin(A-B) = \sin^2 A - \sin^2 B$

(*ii*) $\cos(A+B)\cos(A-B) = \cos^2 A - \cos^2 B$

(J. & K. U. 1958)

(*i*) $\text{Sin}(A+B)\,\text{Sin}(A-B) = (\sin A\cos B + \cos A \sin B)\times$
$(\text{Sin } A \text{ Cos } B - \text{Cos } A \text{ Sin } B)$

$= \sin^2 A\cos^2 B - \cos^2 A \sin^2 B$

$= \sin^2 A(1-\sin^2 B)\ (1-\sin^2 A)\sin^2 B$

$= \sin^2 A - \sin^2 A\sin^2 B - \sin^2 B + \sin^2 A\,\text{Sin}^2 B$

$= \sin^2 A - \sin^2 B.$

(*ii*) $\cos(A+B)\cos(A-B) = (\cos A\cos B - \sin A \sin B)\times$
$(\cos A\cos B + \sin A\sin B)$

$= \cos^2 A\cos^2 B - \sin^2 A\sin^2 B$

$= \cos^2 A(1-\sin^2 B) - (1-\cos^2 A)\sin^2 B$

$= \cos^2 A - \cos^2 A\sin^2 B - \sin^2 B + \cos^2 A\sin^2 B$

$= \cos^2 A - \sin^2 B.$

Ex. 1. Find $\sin 15^\circ$, $\cos 15^\circ$, $\tan 15^\circ$.

$\sin 15^\circ = \sin(45^\circ - 30^\circ) = \sin 45^\circ\cos 30^\circ - \cos 45^\circ \sin 30^\circ$

$$= \frac{1}{\sqrt{2}}\cdot\frac{\sqrt{3}}{2} - \frac{1}{\sqrt{2}}\cdot\ \tfrac{1}{2} = \frac{\sqrt{3}-1}{2\sqrt{2}}.$$

$\cos 15^\circ = \cos(45^\circ - 30^\circ) = \cos 45^\circ \cos 30^\circ + \sin 45^\circ \sin 30^\circ$

$$= \frac{1}{\sqrt{2}}\cdot\frac{\sqrt{3}}{2} + \frac{1}{\sqrt{2}}\cdot\ \tfrac{1}{2} = \frac{\sqrt{3}+1}{2\sqrt{2}}.$$

$$\tan 15^\circ = \tan(45^\circ - 30^\circ) = \frac{\tan 45^\circ - \tan 30^\circ}{1+\tan 45^\circ \tan 30^\circ}$$

$$=\frac{1-\frac{1}{\sqrt{3}}}{1+\frac{1}{\sqrt{3}}}=\frac{\sqrt{3}-1}{\sqrt{3}+1}=\frac{(\sqrt{3}-1)(\sqrt{3}-1)}{(\sqrt{3}+1)(\sqrt{3}-1)}$$

$$=\frac{3+1-2\sqrt{3}}{3-1}=2-\sqrt{3}.$$

Note :— We can also do the above by taking $15°=60°-45°$

Ex. 2. Show that $\tan 69°+\tan 66°+1=\tan 69° \tan 66°$ (P. U. 1936 S.)

Since $69°+66°=135°$

$\therefore \tan (69°+66°)=\tan 135°=\tan (180°-45°)=-\tan 45° =-1.$

$$\therefore \frac{\tan 69°+\tan 66°}{1-\tan 69° \tan 66°}=-1$$

Cross multiplying, $\tan 69°+\tan 66°=-1+\tan 69° \tan 66°$ or $\tan 69°+\tan 66°+1=\tan 69° \tan 66°$

Ex. 3. Prove that (*i*) $\frac{\sin (A+B)}{\sin (A-B)}=\frac{\tan A+\tan B}{\tan A-\tan B}$

(*ii*) $\frac{\cos (A+B)}{\cos (A-B)}=\frac{1-\tan A \tan B}{1+\tan A \tan B}$

(*iii*) $\tan A \pm \tan B=\frac{\sin (A \pm B)}{\cos A \cos B}$

Sol. (*i*) $\frac{\sin (A+B)}{\sin (A-B)}=\frac{\sin A \cos B+\cos A \sin B}{\sin A \cos B-\cos A \sin B}$

$$=\frac{\frac{\sin A}{\cos A}+\frac{\sin B}{\cos B}}{\frac{\sin A}{\cos A}-\frac{\sin B}{\cos B}}$$ (Dividing the Numr. and Denr. by $\cos A \cos B$)

$$=\frac{\tan A+\tan B}{\tan A-\tan B}$$

(*ii*) $\dfrac{\cos(A+B)}{\cos(A-B)}=\dfrac{\cos A\cos B-\sin A\sin B}{\cos A\cos B+\sin A\sin B}$

$=\dfrac{1-\dfrac{\sin A}{\cos A}\cdot\dfrac{\sin B}{\cos B}}{1+\dfrac{\sin A}{\cos A}\cdot\dfrac{\sin A}{\cos B}}$ Dividing the Numr. and Denr. by $\cos A\cos B$)

$=\dfrac{1-\tan A\tan B}{1+\tan A\tan B}.$

(*iii*) $\tan A\pm\tan B=\dfrac{\sin A}{\cos B}\pm\dfrac{\sin B}{\cos B}$

$=\dfrac{\sin A\cos B\pm\cos A\sin B}{\cos A.\cos B}$

$=\dfrac{\sin(A\pm B)}{\cos A.\cos B)}$

Ex. 4. Show that $\tan 5A-\tan 3A-\tan 2A$

$=\tan 5A\tan 3A\tan 2A.$

$\tan 5A=\tan(3A+2A)$

or $\tan 5A=\dfrac{\tan 3A+\tan 2A}{1-\tan 3A\tan 2A}$

Cross multiplying,

$\therefore\ \tan 5A-\tan 5A\tan 3A\tan 2A=\tan 3A+\tan 2A$

or $\tan 5A-\tan 3A-\tan 2A=\tan 5A\tan 3A\tan 2A.$

Exercise 8.

Prove that :—

1. $\cos(45^\circ+A)=\dfrac{1}{\sqrt{2}}(\cos A-\sin A)$

2. $\dfrac{\sin(A+B)}{\cos A\cos B}=\tan A+\tan B$

3. $\sin 165^\circ=\dfrac{\sqrt{3}+1}{2\sqrt{2}}$

4. $\sin 23^\circ\cos 7^\circ+\cos 23^\circ\sin 7^\circ=\sin 30^\circ$

5. $\sqrt{3}\cos 23^\circ - \sin 23^\circ = 2\cos 53^\circ$
6. $\sin A + \sin(120^\circ + A) + \sin(240^\circ + A) = 0$
7. $\dfrac{\sin(\alpha-\beta)}{\sin\alpha\sin\beta} = \cot\beta - \cot\alpha$
8. $\sin(60^\circ+\theta) - \sin(60^\circ-\theta) = \sin\theta$
9. $\cos(45^\circ+A) + \sin(A-45^\circ) = 0$
10. $\tan(45^\circ+A) - \tan(45^\circ-A) = \dfrac{4\tan A}{1-\tan^2 A}$
11. $\dfrac{\tan 2\theta + \tan\theta}{\tan 2\theta - \tan\theta} = \dfrac{\sin 3\theta}{\sin\theta}$
12. (*i*) $\tan 15^\circ + \tan 30^\circ + \tan 15^\circ \tan 30^\circ = 1$ (P.U.1948)
    (*ii*) $\tan 75^\circ - \tan 30^\circ - \tan 75^\circ \tan 30^\circ = 1$ (D.U.1953)
13. $\tan 8A - \tan 3A - \tan 5A = \tan 3A \tan 5A \tan 8A$ (P. U. 1954)
14. $\dfrac{\cos(A-B)}{\cos(A+B)} = \dfrac{1+\tan A\tan B}{1-\tan A\tan B}$
15. (*i*) $\dfrac{\sin(A-B)}{\cos A\cos B} + \dfrac{\sin(B-C)}{\cos B\cos C} + \dfrac{\sin(C-A)}{\cos C\cos A} = 0$

    (*ii*) $\dfrac{\sin(A-B)}{\sin A\sin B} + \dfrac{\sin(B-C)}{\sin B\sin C} + \dfrac{\sin(C-A)}{\sin C\sin A} = 0$
16. (*i*) If $\cos A = \dfrac{1}{7}$ and $\cos B = \dfrac{13}{14}$, where A and B are acute, show that $A - B = 60^\circ$. (P. U.)

    (*ii*) If $\sin A = \dfrac{1}{\sqrt{10}}$, $\sin B = \dfrac{1}{\sqrt{5}}$ show that

    $A + B = \dfrac{\pi}{4}$ (P. U. 1953)
17. If $\sin A = \dfrac{60}{61}$, $\cos B = \dfrac{40}{41}$, where A and B are acute, find $\sin(A+B)$ and $\cos(A+B)$.
18. If $\tan A = \frac{1}{2}$ and $\tan B = \frac{1}{3}$, show that $A + B = 45^\circ$.

Prove the following :—

19. $\dfrac{\sin(A+B)\sin(A-B)}{\cos^2 A\cos^2 B} = \tan^2 A - \tan^2 B$
20. $1 - \cos(A-B)\cos(A+B) = \sin^2 A + \sin^2 B$

21. $\cos\theta\cos\phi=\cos^2\dfrac{\theta-\phi}{2}-\sin^2\dfrac{\theta+\phi}{2}$ (P. U. 1953)

22. $\sin^2\left(\dfrac{\pi}{4}+\dfrac{\theta}{2}\right)-\sin^2\left(\dfrac{\pi}{4}-\dfrac{\theta}{2}\right)=\sin\theta$

23. $\tan 50^\circ=2\tan 10^\circ+\tan 40^\circ$

[**Hint.** $50^\circ=40+10^\circ$ and $\tan 40^\circ\tan 50^\circ$
$=\tan 40^\circ\cot 40^\circ=1$].

24. (*i*) $\tan 65^\circ=2\tan 40^\circ+\tan 25^\circ$

(*ii*) $2\tan 50^\circ+\tan 20^\circ=\tan 70^\circ$ (J. & K. U. 1956)

25. If $A+B=45^\circ$, show that $(\cot A-1)(\cot B-1)=2$ (P. U.)

26. Show that $\sin^2 75^\circ-\sin^2 15=\sin 60^\circ$

27. Prove that $\tan(A+B)\tan(A-B)=\dfrac{\sin^2A-\sin^2B}{\cos^2A-\sin^2B}$

$$=\frac{\cos^2B-\cos^2A}{\cos^2B-\sin^2A}$$

28. If $\tan\theta=\dfrac{m_1-m_2}{1+m_1m_2}$, find m_2; given $m_1=\frac{1}{2}$ and $\tan\theta=\dfrac{2}{9}$.

29. $\sin(n+1)x\cos(n-1)x-\sin 2x=$
$\sin(n-1)x.\cos(n+1)x.$

30. If $\theta+\phi=\alpha$ and $\tan\theta=\lambda\tan\phi$, prove that

$$\tan(\theta-\phi)=\frac{\lambda-1}{\lambda+1}\sin\alpha.$$

31. $\dfrac{\cos 45^\circ+\sin 75^\circ}{\sin 45^\circ-\cos 75^\circ}=\dfrac{\cos 75^\circ+\sin 45^\circ}{\sin 75^\circ+\cos 45^\circ}=\cot 15^\circ$

32. Prove that :—

$\sin 17^\circ 26'\cos 12^\circ 34'+\sin 72^\circ 34'\sin 12^\circ 34'=\frac{1}{2}.$

(J. & K. U. 1955)

4. Trigonometrical functions of $(A+B+C)$.

To expand (*i*) $\sin(A+B+C)$
(*ii*) $\cos(A+B+C)$
(*iii*) $\tan(A+B+C)$

in terms of T-ratios of A,B,C.

(*i*) $\sin(A+B+C)=\sin(\overline{A+B}+C)$
$=\sin(A+B)\cos C+\cos(A+B)\sin C$
$=(\sin A\cos B+\cos A\sin B)\cos C$
$+(\cos A\cos B-\sin A\sin B)\sin C$
$=\sin A\cos B\cos C+\sin B\cos C\cos A$
$+\sin C\cos A\cos B-\sin A\sin B\sin C$

(*ii*) $\cos(A+B+C)=\cos(\overline{A+B}+C)$
$=\cos(A+B)\cos C-\sin(A+B)\sin C$
$=(\cos A\cos B-\sin A\sin B)\cos C$
$-(\sin A\cos B+\cos A\sin B)\sin C$
$=\cos A\cos B\cos C-\cos A\sin B\sin C$
$-\cos B\sin C\sin A-\cos C\sin A\sin B$

(*iii*) $\tan(A+B+C)=\tan(\overline{A+B}+C)=\dfrac{\tan(A+B)+\tan C}{1-\tan(A+B)\tan C}$

$$=\frac{\dfrac{\tan A+\tan B}{1-\tan A\tan B}+\tan C}{1-\dfrac{\tan A+\tan B}{1-\tan A\tan B}\tan C}$$

$$=\frac{\tan A+\tan B+\tan C-\tan A\tan B\tan C}{1-\tan A\tan B-\tan B\tan C-\tan C\tan A}.$$

Ex. 1. If $A+B+C=\pi$, prove that $\tan A+\tan B+\tan C$
$=\tan A\tan B\tan C$ (J. & K. U. 1950)
(P. U. 1951)

1st Method. $\because A+B+C=\pi$
$\therefore \tan(A+B+C)=\tan\pi=0$

or, $\dfrac{\tan A+\tan B+\tan C-\tan A\tan B\tan C}{1-\tan A\tan B-\tan B\tan C-\tan C\tan A}=0$

or, $\tan A+\tan B+\tan C-\tan A\tan B\tan C=0$

or, $\tan A+\tan B+\tan C=\tan A\tan B\tan C$

Second Method. $\because A+B+C=\pi$
$\therefore A+B=\pi-C$
$\therefore \tan(A+B)=\tan(\pi-C)$

or $\dfrac{\tan A+\tan B}{1-\tan A \tan B}=-\tan C$

or $\tan A+\tan B=-\tan C+\tan A \tan B \tan C$

or $\tan A+\tan B+\tan C=\tan A \tan B \tan C$.

5. To express $a \cos\theta+b \sin\theta$ as (*i*) a single sine or (*ii*) a single cosine

(*i*) Put $\left.\begin{array}{l} a=r\sin\alpha \\ b=r\cos\alpha \end{array}\right\}$ where r and α are to be determined.

Squaring and adding $r^2=a^2+b^2$ *i. e.* $r=\sqrt{a^2+b^2}$

Dividing, $\tan\alpha=\dfrac{a}{b}$ *i. e.* $\alpha=\tan^{-1}\dfrac{a}{b}$

Now, $a\cos\theta+b\sin\theta=r\sin\alpha\cos\theta+r\cos\alpha\sin\theta$

$$=r(\sin\alpha\cos\theta+\cos\alpha\sin\theta)$$
$$=r\sin(\alpha+\theta)=r\sin(\theta+\alpha)$$
$$=\sqrt{a^2+b^2}\sin\left(\theta+\tan^{-1}\frac{a}{b}\right)$$

(*ii*) Put $a=r\cos\alpha$

$b=r\sin\alpha$

$\therefore$ as above, $r=\sqrt{a^2+b^2}$ and $\tan\alpha=\dfrac{b}{a}$

Now, $a\cos\theta+b\sin\theta=r(\cos\alpha\cos\theta+\sin\alpha\sin\theta)$

$$=r\cos(\theta-\alpha)$$
$$=\sqrt{a^2+b^2}\cos\left(\theta-\tan^{-1}\frac{b}{a}\right)$$

Note. It is customary to write $\tan\alpha=\dfrac{a}{b}$ or $\dfrac{b}{a}$

in cases (*i*) and (*ii*) above, but that does not always give the correct value of α. The correct value of α is that which satisfies both the equations, in each of the above cases, simultaneously.

Exercise 9.

1. Express $\cos(A-B+C)$ in terms of sines and cosines of A, B, C. (P. U.)

2. If $A+B+C=180°$, show that :—

(i) $\cos A \cos B \cos C=\cos A \sin B \sin C$
$+\cos B \sin C \sin A+\cos C \sin A \sin B-1$

(ii) $\cot A \cot B+\cot B \cot C+\cot C \cot A=1$

3. If $A+B+C=\frac{\pi}{2}$, show that

$\tan A \tan B+\tan B \tan C+\tan C \tan A=1$

(J. & K. U. 1957)

4. Express $\sqrt{3}\cos\theta+\sin\theta$ in terms of the cosine of a single angle and hence find its greatest value. (P. U. 1950)

5. Express (i) $\sqrt{3}\sin\theta+\cos\theta$ and (ii) $\sin\theta+\cos\theta$ in terms of the sine of a single angle.

6. **Double Angles.**

(i) $\sin 2A=\sin(A+A)=\sin A\cos A+\cos A\sin A$

$$=2\sin A\cos A \quad \text{......(Form I)}$$

$$=\frac{2\sin A\cos A}{1}=\frac{2\sin A\cos A}{\cos^2A+\sin^2A}$$

$$=\frac{\dfrac{2\sin A\cos A}{\cos^2 A}}{\dfrac{\cos^2A+\sin^2A}{\cos^2A}}=\frac{2\tan A}{1+\tan^2A}.$$

(ii) $\cos 2A=\cos(A+A)=\cos A\cos A-\sin A\sin A$

$$=\cos^2A-\sin^2A \quad \text{.....(Form I)}$$
$$=1-\sin^2A-\sin^2A$$
$$=1-2\sin^2A. \quad \text{......(Form II)}$$

Also $\cos 2A=\cos^2A-\sin^2A=\cos^2A-(1-\cos^2A)$

$$=2\cos^2 A-1 \quad \text{.....(Form III)}$$

Again $\cos 2A=\dfrac{\cos^2A-\sin^2A}{1}=\dfrac{\cos^2A-\sin^2A}{\cos^2A+\sin^2A}$

$$=\frac{\dfrac{\cos^2 A-\sin^2 A}{\cos^2 A}}{\dfrac{\cos^2 A+\sin^2 A}{\cos^2 A}}=\frac{1-\tan^2 A}{1+\tan^2 A}$$

......(Form IV

Cor. Forms II and III give two very useful results :-

(*a*) $1-\cos 2A=2\sin^2 A$.

(*b*) $1+\cos 2A=2\cos^2 A$.

(*iii*) $\tan 2A=\tan(A+A)=\dfrac{\tan A+\tan A}{1-\tan A\ \tan A}=\dfrac{2\tan A}{1-\tan^2 A}$

Or thus :—We can prove the above results geometrically also by changing B into A in the proof of Art 1.

7. Trigonometric ratios of 3 A.

$\sin 3A=\sin(A+2A)=\sin A\cos 2A+\cos A\sin 2A$

$=\sin A.\ (1-2\sin^2 A)+\cos A.\ 2\sin A\cos A$

(Putting values of sin 2A, cos 2A from Art. 6)

$=\sin A-2\sin^3 A+2\sin A\,(1-\sin^2 A)$

$=3\sin A-4\sin^3 A.$

$\cos 3A=\cos(A+2A)=\cos A\cos 2A-\sin A\sin 2A$

$=\cos A\,(2\cos^2 A-1)-\sin A.\ 2\sin A\cos A$

(Putting values of sin 2A and cos 2A)

$=2\cos^3 A-\cos A-2\cos A\,(1-\cos^2 A)$

$=4\cos^3 A-3\cos A$

$$\tan 3A=\tan(A+2A)=\frac{\tan A+\tan 2A}{1-\tan A\tan 2A}$$

$$=\frac{\tan A+\dfrac{2\tan A}{1-\tan^2 A}}{1-\tan A.\dfrac{2\tan A}{1-\tan^2 A}}=\frac{\tan A-\tan^3 A+2\tan A}{1-\tan^2 A-2\tan^2 A}$$

$$=\frac{3\tan A-\tan^3 A}{1-3\tan^2 A}.$$

Ex. 1. Prove tant $\frac{1+\sin 2\theta}{1-\sin 2\theta}=\tan^2\left(\frac{\pi}{4}+0\right)$

$$\text{L. H. S.}=\frac{1+\frac{2\tan\theta}{1+\tan^2\theta}}{1-\frac{2\tan\theta}{1+\tan^2\theta}}=\frac{1+\tan^2\theta+2\tan\theta}{1+\tan^2\theta-2\tan 0}$$

$$=\frac{(1+\tan\theta)^2}{(1-\tan\theta)^2}=\left(\frac{1+\tan 0}{1-\tan\theta}\right)^2$$

$$\text{R. H. S.}=\frac{\left(\tan\frac{\pi}{4}+\tan\theta\right)^2}{\left(\tan\frac{\pi}{4}-\tan\theta\right)^2}=\left(\frac{1+\tan\theta}{1-\tan\theta}\right)^2$$

$\therefore$ L. H. S.=R. H. S.

Ex. 2. Show that cosec $2A-\cot 2A=\tan A$. (D. U. 1938)

$$\text{L. H. S.}=\frac{1}{\sin 2A}-\frac{\cos 2A}{\sin 2A}=\frac{1-\cos 2A}{\sin 2A}$$

$$=\frac{1-(1-2\sin^2 A)}{2\sin A\cos A}=\frac{2\sin^2 A}{2\sin A\cos A}$$

$$=\tan A$$

Ex. 3. Prove that $2\cos\theta=\sqrt{2+\sqrt{2+2\cos 4\theta}}$

(P. U. 1936)

$$\text{R. H. S.}=\sqrt{2+\sqrt{2+2(2\cos^2 2\theta-1)}}$$

$$=\sqrt{2+\sqrt{2+4\cos^2 2\theta-2}}$$

$$=\sqrt{2+2\cos 2\theta}=\sqrt{2+2(2\cos^2\theta-1)}$$

$$\sqrt{2+4\cos^2\theta-2}=2\cos\theta.$$

Ex. 4 Prove that $\frac{\sin 3\theta+\cos 3\theta}{\cos 0-\sin\theta}=1+\sin 2\theta$

$$\text{L. H. S.}=\frac{3\sin\theta-4\sin^3\theta+4\cos^3\theta-3\cos\theta}{\cos\theta-\sin\theta}$$

$$=\frac{4(\cos^3\theta-\sin^3\theta)-3(\cos\theta-\cos\theta)}{\cos\theta-\sin\theta}$$

$$=\frac{(\cos\theta-\sin\theta)[4(\cos^2\theta+\sin^2\theta+\cos\theta\sin 0)-3]}{\cos\theta-\sin\theta}$$

$$=4(1+\sin\theta\cos\theta)-3$$
$$=1+4\sin\theta\cos\theta=1+2\cdot 2\sin\theta\cos\theta$$
$$=1+2\sin 2\theta$$

Ex. 5. Prove that $\sin A \sin(60^\circ-A)\sin(60^\circ+A)=\frac{1}{4}\sin 3A$

$$\text{L. H. S.}=\sin A(\sin^2 60^\circ-\sin^2 A)$$
$$=\sin A\left(\tfrac{3}{4}-\sin^2 A\right)$$
$$=\sin A\left(\frac{3-4\sin^2 A}{4}\right)=\frac{3\sin A-4\sin^3 A}{4}$$
$$=\frac{\sin 3A}{4}=\frac{1}{4}\sin 3A.$$

Exercise 10

1. If $\sin A=\frac{1}{7}$, find $\cos 2A$.
2. If $\cos A=\frac{3}{5}$, find $\cos 2A$ and $\sin 2A$ (A being acute).
3. Evaluate $\sin 2A$ and $\cos 2A$ if $\sin A=\frac{12}{13}$ (A being acute).
4. Find the values of $\sin 2A$, $\cos 2A$ and $\tan 2A$ when $\tan A=\frac{1}{4}$.
5. Find the values of $\sin 3A$ and $\cos 3A$ when $\sin A=\frac{3}{5}$ (A being acute)

Prove that :—

6. $\dfrac{1-\cos 2A}{1+\cos 2A}=\tan^2 A$

7. (*i*) $\dfrac{\sin 2A}{1+\cos 2A}=\tan A$ (*ii*) $\dfrac{\sin 2A}{1-\cos 2A}=\cot A$

   (*iii*) $\dfrac{\cos 2\theta}{1+\sin 2\theta}=\tan(45^\circ-\theta)$

   (*iv*) $\dfrac{\cos 2\theta}{1-\sin 2\theta}=\cot(45^\circ-\theta)$

8. (*i*) $\cot A-\tan A=2\cot 2A$. (P. U. 1943 S)

   (*ii*) $\cot A+\tan A=2\operatorname{cosec} 2A$ (C. U.)

9. $\dfrac{\cot A-\tan A}{\cot A+\tan A}=\cos 2A$ (C. U.)

10. $\dfrac{1-\cos 2\theta+\sin 2\theta}{1+\cos 2\theta+\sin 2\theta}=\tan \theta$ (C. U.)

11. $\operatorname{cosec} 2A=\tan A+\cot 2A$ (P. U. 1954)

12. $\sec (45°-A) \sec (45°+A)=2 \sec 2A$ (P. U. 1941)

13. $\cos \alpha \cos (60°+\alpha) \cos (60°-\alpha)=\frac{1}{4} \cos 3\alpha.$

14. $\dfrac{\sin 3A}{\sin A}-\dfrac{\cos 3A}{\cos A}=2$

15. $\dfrac{\cos A-\cos 3A}{\sin 3A-\sin A}=\tan 2A$

16. (*i*) $\tan A \tan (120°+A) \tan (120°-A)=\tan 3A$

(*ii*) $\tan A+(\tan 60°+A)+\tan (120°+A)=3 \tan 3A$

17. (*i*) $\sin 4A=4 \sin A \cos^3 A-4 \cos A \sin^3 A$

(*ii*) $\cos 4A=1-8 \cos^2 A+8 \cos^4 A$

[Hint. Put $4A=2.2\,A$]

18. $\dfrac{2 \cos 2A+1}{2 \cos 2A-1}=\tan (60°+A) \tan (60°-A)$

(J. & K U. 1953)

19. $\dfrac{1-\tan^2\left(\dfrac{\pi}{4}+A\right)}{1+\tan^2\left(\dfrac{\pi}{4}+A\right)}=-\sin 2A$

20. $\dfrac{1-\tan \theta \tan 2\theta}{1+\tan \theta \tan 2\theta}=1-4 \sin^2\theta$ (J. & K. U. 1949)

21. $\dfrac{1+\sin 2\theta}{1-\sin 2\theta}=\tan^2\left(\dfrac{\pi}{4}+\theta\right)$ (D. U. 1948)

22. If $\tan \theta=\dfrac{b}{a}$, find the value of $a \cos 2\theta+b \sin 2\theta$

(C. U. 1938)

23. $\dfrac{\tan 5\theta+\tan 3\theta}{\tan 5\theta-\tan 3\theta}=4 \cos 2\theta \cos 4\theta$

24. $\dfrac{1}{1-\tan\theta}-\dfrac{1}{1+\tan\theta}=\tan 2\theta$

25. $(\cos A-\cos B)^2+(\sin A-\sin B)^2=4\sin^2\dfrac{A-B}{2}$

26. If $a\sin A=b\cos A$, prove that

$$\sqrt{\frac{a-b}{a+b}}+\sqrt{\frac{a+b}{a-b}}=\frac{2\cos A}{\sqrt{\cos 2A}}$$

8. Trigonometric ratios of 18° and 72°. (K. U. 1955)

Let $18°=\theta$,

$\therefore\ 5\theta=90°$

or $2\theta=90°-3\theta$

$\therefore\ \sin 2\theta=\sin(90°-3\theta)=\cos 3\theta$

$\therefore\ 2\sin\theta\cos\theta=4\cos^3\theta-3\cos\theta$

Dividing out by $\cos\theta$ (which is not zero), we have

$2\sin\theta=4\cos^2\theta-3$

$\quad=4(1-\sin^2\theta)-3$

i. e. $4\sin^2\theta+2\sin\theta-1=0$

$$\therefore\ \sin\theta=\frac{-2\pm\sqrt{4+16}}{8}=\frac{-1\pm\sqrt{5}}{4}$$

As $\theta=18°$ is an acute angle, $\sin\theta$ must be positive.

Hence $\sin 18°=\dfrac{\sqrt{5}-1}{4}$

Cor. 1. $\cos 18°=\sqrt{1-\sin^2 18°}=\sqrt{1-\left(\dfrac{\sqrt{5}-1}{4}\right)^2}$

$$=\frac{\sqrt{10+2\sqrt{5}}}{4}.$$

From $\sin 18°$ and $\cos 18°$ the remaining T-ratios can be obtained.

Cor. 2. $\sin 72°=\sin(90°-18°)=\cos 18°=\dfrac{\sqrt{10+2\sqrt{5}}}{4}$

Cor. 3. $\cos 72° = \cos (90° - 18°) = \sin 18° = \frac{\sqrt{5}-1}{4}$

9. Trigonometric ratios of 36° and 54°.

Put $\theta = 36°$

$\therefore 5\theta = 180°$

$\therefore 2\theta = 180° - 3\theta$

$\therefore \sin 2\theta = \sin (180° - 3\theta) = \sin 3\theta$

or $2 \sin \theta \cos \theta = 3 \sin \theta - 4 \sin^3\theta$

Dividing out by $\sin \theta$ (which is not zero), we get

$2 \cos\theta = 3 - 4 \sin^2\theta$

$= 3 - 4 (1 - \cos^2\theta)$

or $4 \cos^2\theta - 2 \cos \theta - 1 = 0$

$$\therefore \cos \theta = \frac{2 \pm \sqrt{4+16}}{8} = \frac{1 \pm \sqrt{5}}{4}.$$

But $\theta = 36°$ is an acute angle, hence $\cos \theta$ must be positive.

$$\therefore \cos 36° = \frac{\sqrt{5}+1}{4}.$$

Or thus : $\cos 36° = 1 - 2 \sin^2 18° = 1 - 2 \left(\frac{\sqrt{5}-1}{4}\right)^2$

$$= \frac{\sqrt{5}+1}{4}.$$

Cor. 1. $\sin 36° = \sqrt{1 - \cos^2 36°} = \sqrt{1 - \left(\frac{\sqrt{5}+1}{4}\right)^2}$

$$= \frac{\sqrt{10 - 2\sqrt{5}}}{4}$$

From sin 36° and cos 36° the remaining T-ratios can be obtained.

Cor. 2. $\sin 54° = \sin (90° - 36°) = \cos 36° = \frac{\sqrt{5}+1}{4}$

Cor. 3. $\cos 54° = \cos (90° - 36°) = \sin 36° = \frac{\sqrt{10 - 2\sqrt{5}}}{4}$

Exercise 11

1. Find the values of (*i*) Sec 36°, (*ii*) cosec 10° (P. U.)

Prove that :—

2. $\cos 36° \cos 72° \cos 108° \cos 144° = \frac{1}{16}$
3. $\cos 36° - \sin 18° = \frac{1}{2}$
4. $\sin 162° + \sin 30° = \cos 36°$
5. $\sin \frac{\pi}{5} \sin \frac{2\pi}{5} \sin \frac{3\pi}{5} \sin \frac{4\pi}{5} = \frac{5}{16}$

CHAPTER VII

Submultiple angles

1. To find the T-ratios of A in terms of T-ratios of $A/_2$.
We know that :—

(*i*) $\sin 2A = 2 \sin A \cos A$

(*ii*) (*a*) $\cos 2A = \cos^2 A - \sin^2 A$

(*b*) $\cos 2A = 1 - 2 \sin^2 A$

(*c*) $\cos 2A = 2 \cos^2 A - 1$

(*iii*) $\tan 2\ A = \dfrac{2 \tan A}{1 - \tan^2 A}$

(*iv*) $\sin 2\ A = \dfrac{2 \tan A}{1 + \tan^2 A}$ (*v*) $\cos 2A = \dfrac{1 - \tan^2 A}{1 + \tan^2 A}$

In the above, changing A to $\frac{A}{2}$ (and hence 2A to A), we get

(*i*) $\sin A = 2 \sin \frac{A}{2} \cos A/2$

(*ii*) (*a*) $\cos\ A = \cos^2 \frac{A}{2} - \sin^2\ A/2$

(*b*) $\cos A = 1 - 2 \sin^2 A/2$

(*c*) $\cos A = 2\ \cos^2 \frac{A}{2} - 1$

(*iii*) $\tan\ A = \dfrac{2 \tan A/_2}{1 - \tan^2 A/_2}$

(*iv*) $\sin\ A = \dfrac{2 \tan A/_2}{1 + \tan^2 A/_2}$ (*v*) $\cos\ A = \dfrac{1 - \tan^2 A/_2}{1 + \tan^2 A/_2}$

Note :—The student is advised to deduce the formula (*iv*) and (*v*) independently as in Article 6. of Chapter (*vi*).

Exercise 12.

Prove that :—

1. (*i*) $\sin A = \dfrac{2 \tan A/_2}{1 + \tan^2 A}$, (*ii*) $\cos A = \dfrac{1 - \tan^2 A/_2}{1 + \tan^2 A/_2}$

(J. & K. U. 1961)

2. (*i*) $\dfrac{1 - \cos A}{\sin A} = \tan A/_2$, (*ii*) $\dfrac{1 + \cos A}{\sin A} = \cot A/_2$

3. $\tan\left(\dfrac{\pi}{4} + \dfrac{A}{2}\right) = \sec A + \tan A$ (P. U.)

4. $\tan\left(\dfrac{\pi}{4} - \dfrac{A}{2}\right) = \sqrt{\dfrac{1 - \sin A}{1 + \sin A}} = \sec A - \tan A$

(P. U.)

5. $\operatorname{cosec} A + 2 \operatorname{cosec} 2A = \sec A \cot A/_2$

(J. & K. U. 1952)

6. $1 + \tan A \tan \dfrac{A}{2} = \tan A \cot \dfrac{A}{2} - 1 = \sec A$

(J. &. K. U. 1961)

2. To express $\sin \dfrac{A}{2}$, $\cos \dfrac{A}{2}$ and $\tan \dfrac{A}{2}$ in terms of $\cos A$.

We know that, $\cos A = 1 - 2 \sin^2 \dfrac{A}{2}$

Transposing, $2 \sin^2 \dfrac{A}{2} = 1 - \cos A$

$$\therefore \sin \frac{A}{2} = \pm \sqrt{\frac{1 - \cos A}{2}} \qquad \text{......(1)}$$

Again, because

$\cos A = 2 \cos^2 \dfrac{A}{2} - 1$

Transposing, $2 \cos^2 \dfrac{A}{2} = 1 + \cos A$

$$\therefore \cos \frac{A}{2} = \pm \sqrt{\frac{1 + \cos A}{2}} \qquad \text{......(2)}$$

Dividing (1) by (2), $\tan\dfrac{A}{2}=\pm\sqrt{\dfrac{1-\cos A}{1+\cos A}}$ ......(3)

Note. The sign to be attached with the radicals in (1), (2) and(3) will depend on the quadrant in which $\dfrac{A}{2}$ lies.

3. To express $\sin\dfrac{A}{2}$, $\cos\dfrac{A}{2}$ and $\tan\dfrac{A}{2}$ in terms of sin A.

We know that $2\sin\dfrac{A}{2}\cos\dfrac{A}{2}=\sin A$ ..... (1)

But $\sin^2\dfrac{A}{2}+\cos^2\dfrac{A}{2}=1$ ......(2)

Adding (1) and (2),

$$\left[\sin\frac{A}{2}+\cos\frac{A}{2}\right]^2=1+\sin A$$

or $\sin\dfrac{A}{2}+\cos\dfrac{A}{2}=\pm\sqrt{1+\sin A}$ ......(3)

Subtracting (1) from (2),

$$\left[\sin\frac{A}{2}-\cos\frac{A}{2}\right]^2=1-\sin A$$

or $\sin\dfrac{A}{2}-\cos\dfrac{A}{2}=\pm\sqrt{1-\sin A}$ ......(4)

From (3) and (4) by addition,

$$2\sin\frac{A}{2}=\pm\sqrt{1+\sin A}\pm\sqrt{1-\sin A}$$

$$\therefore \sin\frac{A}{2}=\frac{1}{2}\{\pm\sqrt{1+\sin A}\pm\sqrt{1-\sin A}\}$$

Similarly from (3) and (4) by subtraction, we get

$$\cos\frac{A}{2}=\frac{1}{2}\{\pm\sqrt{1+\sin A}\mp\sqrt{1-\sin A}\}$$
$$\ldots\ldots(4)$$

Dividing (5) by (6),

$$\tan\frac{A}{2}=\frac{\pm\sqrt{1+\sin A}\pm\sqrt{1-\sin A}}{\pm\sqrt{1+\sin A}\mp\sqrt{1-\sin A}}$$

Note : The signs of the radicals in (3) and (4) must be determined before adding or subtracting them.

To determine the signs in (3) and (4) we proceed as below :—

$$\sin\frac{A}{2}+\cos\frac{A}{2}=\sqrt{2}\left[\frac{1}{\sqrt{2}}\sin\frac{A}{2}+\frac{1}{\sqrt{2}}\cos\frac{A}{2}\right]$$
$$=\sqrt{2}\left[\cos 45^\circ\sin\frac{A}{2}+\sin 45^\circ\cos\frac{A}{2}\right]$$
$$=\sqrt{2}\sin\left[\frac{A}{2}+45^\circ\right]$$

Hence sign of $\sin\frac{A}{2}+\cos\frac{A}{2}$ is the same as the sign of $\sin\left[45^\circ+\frac{A}{2}\right]$.

Similarly $$\sin\frac{A}{2}-\cos\frac{A}{2}=\sqrt{2}\left[\frac{1}{\sqrt{2}}\sin\frac{A}{2}-\frac{1}{\sqrt{2}}\cos\frac{A}{2}\right]$$
$$=\sqrt{2}\sin\left[\frac{A}{2}-45^\circ\right]$$

Hence the sign of $\sin\frac{A}{2}-\cos\frac{A}{2}$ is the same as the sign of $\sin\left[\frac{A}{2}-45^\circ\right]$.

Or thus :—We know that $\sin\theta$ is $+ve$ in the quadrants I and II and negative in the quadrants III and IV. Applying this rule to $\sin\left[\frac{A}{2}+45^\circ\right]$ and

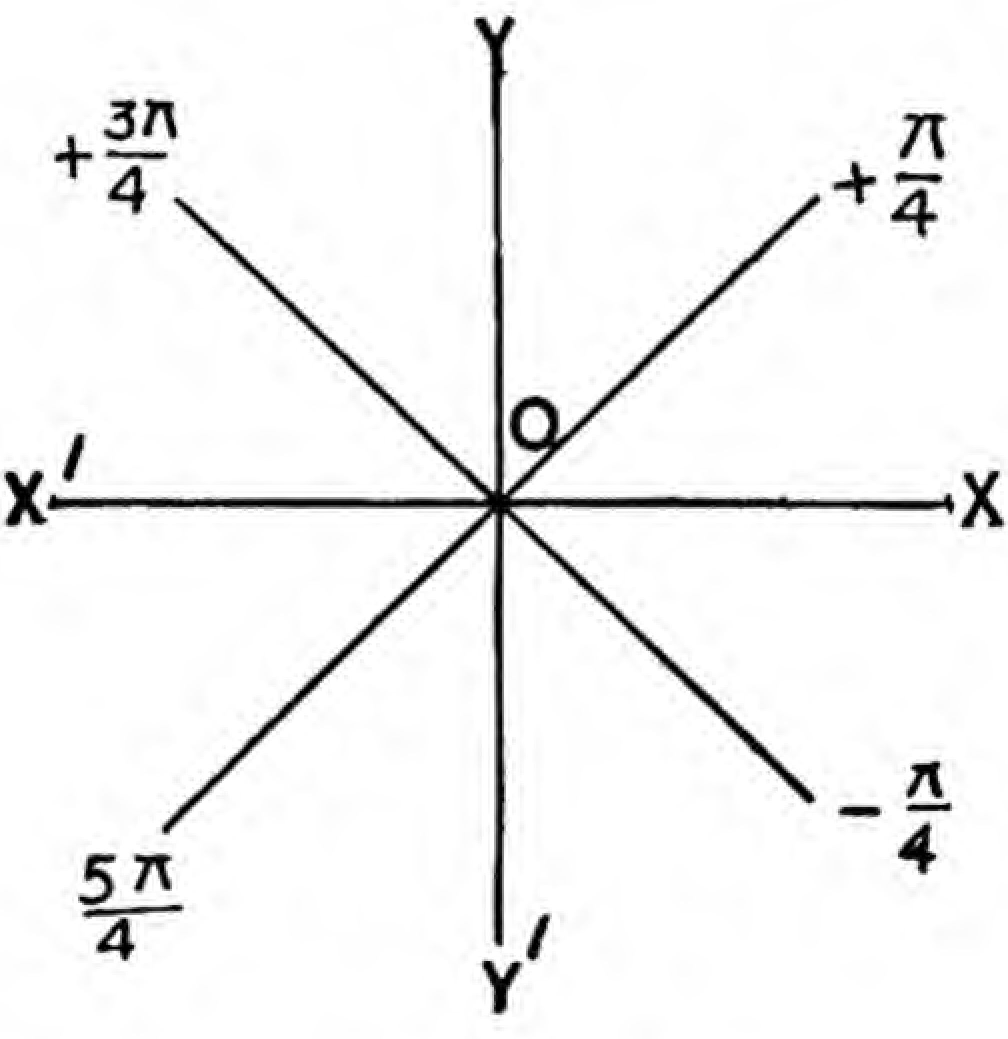
$\sin\left[\frac{A}{2} - 45°\right]$ we can easily deduce the following :—

I. If $\frac{A}{2}$ lies between $\frac{-\pi}{4}$ and $\frac{\pi}{4}$,

$\sin\frac{A}{2} + \cos\frac{A}{2}$ is positive and $\sin\frac{A}{2} - \cos\frac{A}{2}$ is negative.

II. If $\frac{A}{2}$ lies between $\frac{\pi}{4}$ and $\frac{3\pi}{4}$,

$\sin\frac{A}{2} + \cos\frac{A}{2}$ is positive and $\sin\frac{A}{2} - \cos\frac{A}{2}$ is also positive.



II. If $\frac{A}{2}$ lies between $\frac{3\pi}{4}$ and $\frac{5\pi}{4}$,

$\sin\frac{A}{2} + \cos\frac{A}{2}$ is negative and $\sin\frac{A}{2} - \cos\frac{A}{2}$ is positive.

IV. If $\frac{A}{2}$ lies between $\frac{5\pi}{4}$ and $\frac{-\pi}{4}\left[i.e. \frac{7\pi}{4}\right]$

$\sin\frac{A}{2}+\cos\frac{A}{2}$ is negative and $\sin\frac{A}{2}-\cos\frac{A}{2}$ is also negative.

4. To express $\tan\frac{A}{2}$ in terms of $\tan A$.

$$\because \tan A=\frac{2\tan\frac{A}{2}}{1-\tan^2\frac{A}{2}}$$

$$\therefore \tan A-\tan A.\tan^2\frac{A}{2}=2\tan\frac{A}{2}$$

Arranging it as a quadratic in $\tan\frac{A}{2}$, we get

$$\tan A.\tan^2\frac{A}{2}+2\tan\frac{A}{2}-\tan A=0$$

$$\therefore \tan\frac{A}{2}=\frac{-2\pm\sqrt{4+4\tan^2 A}}{2\tan A}=\frac{-1\pm\sqrt{1+\tan^2 A}}{\tan A}$$

The ambiguity of sign can be removed when the value of $\frac{A}{2}$ is known.

Ex. 1. Find the values of $\sin 22\frac{1}{2}°$, $\cos 22\frac{1}{2}°$ and $\tan 22°\frac{1}{2}$ (P. U. 1934)

$$\sin 22\tfrac{1}{2}°=\pm\sqrt{\frac{1-\cos 45°}{2}}\quad\left[\text{by Art. 10, putting } \frac{A}{2}=22\tfrac{1}{2}°\right]$$

$$=\pm\sqrt{\frac{1-\frac{1}{\sqrt{2}}}{2}}=\pm\sqrt{\frac{\sqrt{2}-1}{2\sqrt{2}}}$$

$$=\pm\sqrt{\frac{2-\sqrt{2}}{4}}=\pm\frac{\sqrt{2-\sqrt{2}}}{2} \qquad \text{..........(1)}$$

$$\text{Again,}\cos 22\tfrac{1}{2}^\circ = \pm\sqrt{\frac{1+\cos 45^\circ}{2}}$$

$$=\pm\sqrt{\frac{\sqrt{2}+1}{2\sqrt{2}}} = \pm\sqrt{\frac{2+\sqrt{2}}{2}} = \pm\frac{\sqrt{2+\sqrt{2}}}{2} \quad \ldots\ldots(2)$$

As $22\frac{1}{2}^\circ$ is acute, $\sin 22\frac{1}{2}^\circ$ and $\cos 22\frac{1}{2}^\circ$ are both positive.

Dividing (1) and (2), $\tan 22\frac{1}{2}^\circ = \sqrt{\dfrac{1-\cos 45^\circ}{1+\cos 45^\circ}} = \dfrac{\sqrt{2-\sqrt{2}}}{\sqrt{2+\sqrt{2}}}$

Ex. 2. Find $\sin 9^\circ$ and $\cos 9^\circ$, given that $\sin 18^\circ = \dfrac{\sqrt{5}-1}{4}$

Put $\dfrac{A}{2}=9^\circ$, so that $A=18^\circ$

$\therefore \sin 9^\circ + \cos 9^\circ = \pm\sqrt{1+\sin 18^\circ}$

and $\sin 9^\circ - \cos 9^\circ = \pm\sqrt{1-\sin 18^\circ}$

Now $\sin\left(\dfrac{A}{2}+45^\circ\right) = \sin(9^\circ+45^\circ) = \sin 54^\circ$, which is positive.

and $\sin\left(\dfrac{A}{2}-45^\circ\right) = \sin(9^\circ-45^\circ) = -\sin 36^\circ$, which is negative.

$$\therefore \sin 9^\circ + \cos 9^\circ = +\sqrt{1+\sin 18^\circ} = \sqrt{1+\frac{\sqrt{5}-1}{4}}$$

$$=\frac{\sqrt{3+\sqrt{5}}}{2}$$

$$\sin 9^\circ - \cos 9^\circ = -\sqrt{1-\sin 18^\circ} = -\sqrt{1-\frac{\sqrt{5}-1}{4}}$$

$$=\frac{-\sqrt{5-\sqrt{5}}}{2}$$

Adding and subtracting these,

$$\sin 9^\circ = \frac{\sqrt{3+\sqrt{5}}-\sqrt{5-\sqrt{5}}}{4}$$

and $\cos 9° = \dfrac{\sqrt{3+\sqrt{5}} + \sqrt{5-\sqrt{5}}}{4}$.

Ex. 3. Given $\sin 210° = -\frac{1}{2}$, find the value of $\sin 105°$ and $\cos 105°$.

Putting $\frac{A}{2} = 105°$ and noting that $105°$ lies between $\frac{\pi}{2}$ and $\frac{3\pi}{4}$, we have

$$\sin 105° + \cos 105° = +\sqrt{1+\sin 210°} = +\frac{1}{\sqrt{2}}$$

$$\sin 105° - \cos 105° = +\sqrt{1-\sin 210°} = +\frac{\sqrt{3}}{\sqrt{2}}$$

Adding and subtracting these,

$$\sin 105° = \frac{1+\sqrt{3}}{2\sqrt{2}}$$

and $\cos 105° = \dfrac{1-\sqrt{3}}{2\sqrt{2}}$.

Note :—It is easier to determine the signs by the method illustrated in Ex. 2.

Exercise 13.

1. Find the values of $\sin 165°$, and $\cos 165°$, given that $\cos 330° = \frac{\sqrt{3}}{2}$.

2. Deduce the value of $\tan 15°$ from $\cos 30°$. (P. U.)

3. Given that $\sin 30° = \frac{1}{2}$, find the values of $\sin 15°$ and $\cos 15°$. (P. U. 1949)

4. Show that if A lies between $450°$ and $540°$, then $2 \sin \frac{A}{2} = -\sqrt{1+\sin A} - \sqrt{1-\sin A}$

5. Find the values of $\sin 112°\frac{1}{2}$ and $\cos 112°\frac{1}{2}$ when $\sin 225° = -\frac{1}{\sqrt{2}}$.

6. Find $\tan\frac{A}{2}$, $\sin\frac{A}{2}$, $\cos\frac{A}{2}$ if $\tan A=\frac{21}{20}$

and $\frac{A}{2}$ lies in the first quadrant.

7. If $A=340°$, prove that

$$2\sin\frac{A}{2}=-\sqrt{1+\text{Sin } A}+\sqrt{1-\text{Sin } A}$$

and $$2\cos\frac{A}{2}=-\sqrt{1+\text{Sin } A}-\sqrt{1-\text{Sin } A}$$

. Within what limits must $\frac{A}{2}$ lie if

$$2\sin\frac{A}{2}=-\sqrt{1+\text{Sin } A}+\sqrt{1-\text{Sin } A}$$

(M. U. 1947)

9. Express $\cos 5\theta$ in terms of $\cos\theta$ and hence deduce that $\cos 18°=\frac{\sqrt{10+2\sqrt{5}}}{4}$

(P. U. 1949)

10. (*i*) Find the values of sin 18° and cos 36° and show that they are the roots of equation $4x^2-2\sqrt{5}x+1=0$.

(*ii*) Show that $\cos\frac{\pi}{15}\cos\frac{2\pi}{15}\cos\frac{4\pi}{15}\cos\frac{7\pi}{15}=\frac{1}{16}$

(P. U. 1950)

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CHAPTER VIII

Sum and Product Formulae

1. Products as sums and differences.

We have proved that

$$\sin (A+B)=\sin A \cos B+\cos A \sin B \qquad \text{......}(a)$$

$$\sin (A-B)=\sin A \cos B-\cos A \sin B \qquad \text{.....}(b)$$

Adding (a) and (b),

$$\sin (A+B)+\sin (A-B)=2 \sin A \cos B$$

Subtracting (b) from (a),

$$\sin (A+B)-\sin (A-B)=2 \cos A \sin B$$

$$\therefore\ 2 \sin A \cos B=\sin (A+B)+\sin (A-B) \text{......}(1)$$

$$\text{and } 2 \cos A \sin B=\sin (A+B)-\sin (A-B) \text{......}(2)$$

Again, from $\cos (A+B)=\cos A \cos B-\sin A \sin B$ .....(c)

and $\cos (A-B)=\cos A \cos B+\sin A \sin B$......$(d)$

Adding (c) and (d), $\cos (A+B)+\cos (A-B)=2 \cos A \cos B$

Subtracting (c) from (d), $\cos (A-B)-\cos(A+B)=2 \sin A \sin B$

$$\text{or } \quad 2 \cos A \cos B=\cos (A+B)+\cos (A-B) \text{......}(3)$$

$$\text{and } \quad 2 \sin A \sin B=\cos (A-B)-\cos (A+B) \text{......}(4)$$

These four formulae can be remembered easily thus :—

1. **2 sin . cos = sin (Sum) + sin (diff.)**
2. **2 cos . sin = sin (Sum) − sin (diff.)**
3. **2 cos . cos = cos (Sum) + cos (diff.)**
4. **2 sin . sin = cos (diff.) − cos (sum)**

where by 'diff.' we mean "first angle—second angle".

Note : (*a*) When applying formulae it is convenient to put the larger angle first.

Thus $2 \sin 30^\circ \cos 40^\circ=2 \cos 40^\circ \sin 30^\circ=\sin 70^\circ-\sin 10^\circ$

(*b*) Formula (4) requires special attention. Here 'diff' comes first and 'sum' afterwards.

(*c*) On the R. H. S. of the formulae you have either both sines or both cosines.

2. Sums and differences as products.

In the formulae of art. 1, put $\left.\begin{array}{l} A+B=C \\ A-B=D \end{array}\right\}$

$\therefore$ adding and subtracting, $A=\dfrac{C+D}{2}$ and $B=\dfrac{C-D}{2}$

Substituting these values in each of the equations (1) to (4), we get

$$\sin C+\sin D=2\sin\frac{C+D}{2}\cos\frac{C-D}{2} \quad \ldots\ldots(5)$$

$$\sin C-\sin D=2\cos\frac{C+D}{2}\sin\frac{C-D}{2} \quad \ldots\ldots(6)$$

$$\cos C+\cos D=2\cos\frac{C+D}{2}\cos\frac{C-D}{2} \quad \ldots\ldots(7)$$

$$\cos C-\cos D=-2\sin\frac{C+D}{2}\sin\frac{C-D}{2} \quad \ldots\ldots(8)$$

Formulae (5) to (8) can be remembered thus :—

5. **sin + sin = 2 cos (½ the sum) cos (½ the diff.)**
6. **sin - sin = 2 cos (½ the sum) sin (½ the diff.)**
7. **cos + cos = 2 cos (½ the sum) cos (½ the diff.)**
8. **cos − cos = −2 sin (½ the sum) sin (½ the diff.)**

Note. (*a*) Special attention must be paid to minus sign in (8).

(*b*) Formula (8) can also be written as

$$\mathbf{\cos C-\cos D=2\sin\frac{C+D}{2}\sin\frac{D-C}{2}}$$

Ex. 1. Express as sums or differences :—

(*i*) $2\sin 5\theta\cos 2\theta$ (*ii*) $2\cos 4\theta\sin\theta$

(*iii*) $\cos 3\theta\cos\theta$ (*iv*) $\sin\dfrac{\theta}{2}\sin\dfrac{3\theta}{2}$

Sol. (*i*) $2 \sin 5\theta \cos 2\theta = \sin (5\theta + 2\theta) + \sin (5\theta - 2\theta)$

$$= \sin 7\theta + \sin 3\theta$$

(*ii*) $2 \cos 4\theta \sin \theta = \sin (4\theta + \theta) - \sin (4\theta - \theta) = \sin 5\theta - \sin 3\theta$

(*iii*) $\cos 3\theta \cos \theta = \frac{1}{2} (2 \cos 3\theta \cos \theta)$

$$= \tfrac{1}{2} [\cos (3\theta + \theta) + \cos (3\theta - \theta)]$$

$$= \tfrac{1}{2} (\cos 4\theta + \cos 2\theta)$$

(*iv*) $\sin \frac{\theta}{2} \sin \frac{3\theta}{2} = \frac{1}{2} \left(2 \sin \frac{\theta}{2} \sin \frac{3\theta}{2}\right)$

$$= \frac{1}{2} \left[\cos \left(\frac{\theta}{2} - \frac{3\theta}{2}\right) - \cos \left(\frac{\theta}{2} + \frac{3\theta}{2}\right)\right]$$

$$= \frac{1}{2} \left[\cos (-\theta) - \cos 2\theta \right] = \frac{1}{2} (\cos\theta - \cos 2\theta)$$

Ex. 2. Prove that $4 (\cos 6° + \sin 24°) = \sqrt{3} + \sqrt{15}$

(Agra U.)

Changing cosine into sine of the complementary angle, we have

L.H.S. $= 4[\cos (90° - 84°) + \sin 24°] = 4 [\sin 84° + \sin 24°]$

$$= 4 \left[2 \sin \frac{84° + 24°}{2} \cos \frac{84° - 24°}{2}\right]$$

$$= 8 \sin 54° \cos 30° = 8 \frac{\sqrt{5} + 1}{4} \cdot \frac{\sqrt{3}}{2} = \sqrt{15} + \sqrt{3}$$

[$\because \sin 54° = \cos 36°$]

Ex. 3. Prove that $\frac{\sin \theta + 2 \sin 3\theta + \sin 5\theta}{\sin 3\theta + 2 \sin 5\theta + \sin 7\theta} = \frac{\sin 3\theta}{\sin 5\theta}$

L. H. S. $= \frac{(\sin \theta + \sin 5\theta) + 2 \sin 3\theta}{(\sin 3\theta + \sin 7\theta) + 2 \sin 5\theta}$

$$= \frac{2 \sin 3\theta \cos 2\theta + 2 \sin 3\theta}{2 \sin 5\theta \cos 2\theta + 2 \sin 5\theta}$$

$$= \frac{2 \sin 3\theta (\cos 2\theta + 1)}{2 \sin 5\theta (\cos 2\theta + 1)} = \frac{\sin 3\theta}{\sin 5\theta}$$

Ex. 4. Show that $\sin 20° \sin 40° \sin 60° \sin 80° = \frac{3}{16}$

(J. & K. U. 1949)

$$\text{L.H.S.} = \frac{1}{2}\left[2 \sin 20° \sin 40°\right]\frac{\sqrt{3}}{2}\sin 80°$$

$$= \frac{\sqrt{3}}{4}\left[\cos 20° - \cos 60°\right]\sin 80°$$

$$= \frac{\sqrt{3}}{4}\left[\cos 20° \sin 80° - \cos 60° \sin 80°\right]$$

$$= \frac{\sqrt{3}}{4} \cdot \frac{1}{2}\left[2 \cos 20° \sin 80° - 2 \cos 60° \sin 80°\right]$$

$$= \frac{\sqrt{3}}{8}\left[(\sin 100° + \sin 60°) - 2 \times \tfrac{1}{2} \sin 80°\right]$$

$$= \frac{\sqrt{3}}{8}\left[\sin (180 - 80°) + \frac{\sqrt{3}}{2} - \sin 80°\right]$$

$$= \frac{\sqrt{3}}{8}\left[\sin 80° + \frac{\sqrt{3}}{2} - \sin 80°\right]$$

$$= \frac{\sqrt{3}}{8}\left[\frac{\sqrt{3}}{2}\right] = \frac{3}{16}.$$

Exercise 14.

Express the following as sums or differences of sines and cosines :—

1. $2 \sin 2A \cos A$
2. $2 \cos 5x \cos x$
3. $2 \sin 3A \sin A$
4. $2 \sin x \cos 2x$

Express in the form of a product :—

5. $\sin 6A + \sin 2A$
6. $\cos 3A - \cos A$
7. $\sin 4x - \sin 3x$
8. $\cos x - \cos 2x$
9. $\sin 2x - \sin 5x$
10. $\cos 3x + \cos x$

Prove the following :—

11. (*i*) $\dfrac{\cos 75° + \cos 15°}{\sin 75° + \sin 15°} = \sqrt{3}$

(ii) $\dfrac{\cos 9^\circ+\sin 9^\circ}{\cos 9^\circ-\sin 9^\circ}=\tan 54^\circ$

12. $\dfrac{\sin 5A-\sin 3A}{\cos 3A+\cos 5A}=\tan A$ 13. $\dfrac{\cos\theta-\cos 3\theta}{\sin 3\theta-\sin\theta}=\tan 2\theta$

14. $\dfrac{\cos 2B-\cos 2A}{\sin 2B+\sin 2A}=\tan(A-B)$

15. $\dfrac{\sin A+\sin B}{\cos A+\cos B}=\tan\dfrac{A+B}{2}$

16. $\dfrac{\sin A+\sin B}{\sin A-\sin B}=\dfrac{\tan\dfrac{A+B}{2}}{\tan\dfrac{A-B}{2}}$ (P. U. 1943)

17. $\sin 51^\circ+\cos 81^\circ=\cos 21^\circ$ (P. U.)

18. $\sin 71^\circ-\cos 79^\circ=\cos 41^\circ$

19. $\dfrac{\sin A+\sin 3A+\sin 5A}{\cos A+\cos 3A+\cos 5A}=\tan 3A$

20. $\dfrac{\sin A+\sin 3A+\sin 5A+\sin 7A}{\cos A+\cos 3A+\cos 5A+\cos 7A}=\tan 4A$

(A. U. 1948)

21. $\cos 3A\cos 8A+\sin 4A\sin 7A=\cos A\cos 4A$

22. $\dfrac{\sin 11A\sin A+\sin 7A\sin 3A}{\cos 11A\sin A+\cos 7A\sin 3A}=\tan 8A$ (P. U.)

23. $\dfrac{\sin A+\sin(A+B)+\sin(A+2B)}{\cos A+\cos(A+B)+\cos(A+2B)}=\tan(A+B)$

24. (i) $(\cos\alpha+\cos\beta)^2+(\sin\alpha+\sin\beta)^2$

$=4\cos^2\dfrac{\alpha-\beta}{2}$ (M. U. 1949)

(ii) $\sin 50^\circ-\sin 70^\circ+\sin 10^\circ=0$ (D. U. 1938)

(iii) $\sin 70^\circ-\cos 80^\circ=\cos 40^\circ$ (J. & K. U. 1958)

25. $\cos 20^\circ\cos 40^\circ\cos 60^\circ\cos 80^\circ=\frac{1}{16}$ (J. & K. U. 1957)

26. $\sin 20^\circ\sin 40^\circ\sin 80^\circ\sin 90^\circ=\dfrac{\sqrt{3}}{8}$ (P. U. 1947)

27. (*i*) $\sin 10^\circ \sin 50^\circ \sin 60^\circ \sin 70^\circ = \frac{\sqrt{3}}{16}$ (P. U. 1954)

(*ii*) $\tan 20^\circ \tan 40^\circ \tan 60^\circ \tan 80^\circ = 3$

28. $\sin (54^\circ + A) \sin (54^\circ - A) + \sin (360^\circ - A) \sin (360^\circ + A) = \cos 2A$

29. $\frac{\cos 37^\circ + \sin 37^\circ}{\cos 37^\circ - \sin 37^\circ} = \cot 8^\circ$

30. $\frac{\sin^2 A - \sin^2 B}{\sin A \cos A - \sin B \cos B} = \tan (A+B)$ (J.&K. U. 1960)

31. $\frac{\sin (A-B) + \sin A + \sin (A+B)}{\sin (C-B) + \sin C + \sin (C+B)} = \frac{\sin A}{\sin C}$

32. $\cos \alpha + \cos \left(\alpha + \frac{2\pi}{3}\right) + \cos \left(\alpha + \frac{4\pi}{3}\right) = 0$

33. If $\sin \theta = n \sin (\theta + 2\alpha)$,

show that $\tan (\theta + \alpha) = \frac{1+n}{1-n} \tan \alpha$

34. $\sin 3A + \sin 2A - \sin A = 4 \sin A \cos \frac{A}{2} \cos \frac{3A}{2}$

(J. & K. U. 1955)

35. If $m = \frac{\cos \alpha + \sin \beta}{\cos \beta + \sin \alpha}$, show that :—

$$\frac{1-m}{1+m} = \tan \tfrac{1}{2} (\alpha - \beta)$$

Trigonometrical Identities,

3. When three angles A, B, C satisfy some relation such as $A+B+C=180^\circ$ (as is the case with the angles of a triangle) or $A+B+C=90^\circ$, many interesting identities connecting the trigonometrical ratios of these angles can be proved. Some standard examples illustrating this are given below :—

Ex. 1. If $A+B+C=180^\circ$, prove that

$\sin 2A + \sin 2B + \sin 2C = 4 \sin A \sin B \sin C$.

(J. & K. U. 1955)

We notice that R. H. S. is in factors, hence changing sum into product.

$$\text{L.H.S.} = (\sin 2A + \sin 2B) + \sin 2C$$
$$= 2\sin(A+B)\cos(A-B) + \sin 2C$$

But $A+B+C=180°$, $\therefore A+B=180°-C$

$$\therefore \text{L.H.S.} = 2\sin(180°-C)\cos(A-B) + 2\sin C\cos C$$
$$= 2\sin C\cos(A-B) + 2\sin C\cos C$$
$$= 2\sin C[\cos(A-B) + \cos C]$$
$$= 2\sin C[\cos(A-B) + \cos(180° - \overline{A+B})$$

$$[\because A+B+C=180°, \therefore C=180°-(A+B)]$$

$$= 2\sin C[\cos(A-B) - \cos(A+B)]$$
$$= 2\sin C.\, 2\sin A\sin B = 4\sin A\sin B\sin C$$

Note :—After taking out $\sin C$ as a common factor, we put the remainder of the expression in tems of A, B.

Ex. 2. If $A+B+C=180°$, show that

$$\cos A + \cos B + \cos C = 1 + 4\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$$

$$\text{L. H. S.} = 2\cos\frac{A+B}{2}\cos\frac{A-B}{2} + \cos C$$

$$= \cos\left(90° - \frac{C}{2}\right)\cos\frac{A-B}{2} + \cos C$$

$$\left[\because \frac{A+B}{2} = \frac{180°-C}{2} = 90° - \frac{C}{2}\right]$$

$$= 2\sin\frac{C}{2}\cos\frac{A-B}{2} + \left(1 - 2\sin^2\frac{C}{2}\right)$$

$$= 1 + 2\sin\frac{C}{2}\left[\cos\frac{A-B}{2} - \sin\frac{C}{2}\right]$$

$$= 1 + 2\sin\frac{C}{2}\left[\cos\frac{A-B}{2} - \sin\left(90° - \frac{A+B}{2}\right)\right]$$

$$\left[\because \frac{A+B+C}{2} = 90°, \therefore \frac{C}{2} = 90° - \frac{A+B}{2}\right]$$

$$=1+2\sin\frac{C}{2}\left[\cos\frac{A-B}{2}-\cos\frac{A+B}{2}\right]$$

$$=1+2\sin\frac{C}{2}\left[2\sin\frac{A}{2}\sin\frac{B}{2}\right]$$

$$=1+4\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}.$$

Ex. 3. If $A+B+C=180°$, show that

$$\cot\frac{A}{2}+\cot\frac{B}{2}+\cot\frac{C}{2}=\cot\frac{A}{2}\cot\frac{B}{2}\cot\frac{C}{2}$$

(J. & K. U. 1951)

$$\because A+B+C=180°,\ \therefore \frac{A}{2}+\frac{B}{2}=90°-\frac{C}{2}$$

or $$\cot\left(\frac{A}{2}+\frac{B}{2}\right)=\cot\left(90°-\frac{C}{2}\right)$$

or $$\frac{\cot\frac{B}{2}\cot\frac{A}{2}-1}{\cot\frac{B}{2}+\cot\frac{A}{2}}=\tan\frac{C}{2}$$

or $$\frac{\cot\frac{B}{2}\cot\frac{A}{2}-1}{\cot\frac{B}{2}+\cot\frac{A}{2}}=\frac{1}{\cot\frac{C}{2}}$$

Cross multiplying and transposing terms, we get

$$\cot\frac{A}{2}+\cot\frac{B}{2}+\cot\frac{C}{2}=\cot\frac{A}{2}\cot\frac{B}{2}\cot\frac{C}{2}$$

or else :—Take tangents of both sides of

$$\frac{A}{2}+\frac{B}{2}=90°-\frac{C}{2},\ \text{and}$$

then change into cotangents.

Ex. 4. If $A+B+C=\pi$, prove that

$$\cos^2 A+\cos^2 B-\cos^2 C=1-2\sin A\sin B\cos C$$

$$\begin{aligned}
\text{L. H. S.} &= \tfrac{1}{2}(2\cos^2 A+2\cos^2 B-2\cos^2 C)\\
&= \tfrac{1}{2}[(1+\cos 2A)+(1+\cos 2B)-2\cos^2 C]\\
&= \tfrac{1}{2}[2+(\cos 2A+\cos 2B)-2\cos^2 C]\\
&= \tfrac{1}{2}[2+2\cos(A+B)\cos(A-B)-2\cos^2 C]\\
&= \tfrac{1}{2}[2+2\cos(180^\circ-C)\cos(A-B)-2\cos^2 C]\\
&= \tfrac{1}{2}[2-2\cos C\cos(A-B)-2\cos^2 C]\\
&= \tfrac{1}{2}[1-\cos C\{\cos(A-B)+\cos C\}]\\
&= [1-\cos C\{\cos(A-B)-\cos(A+B)\}]\\
&\qquad \because \cos C=\cos(180^\circ-\overline{A+B})\\
&= 1-\cos C\{2\sin A\sin B\}\\
&= 1-2\sin A\sin B\cos C.
\end{aligned}$$

Note :— When powers of a T-ratio occur, as in $\cos^2 A$ or $\sin^3 A$, it is more convenient to change them into T-ratio of 2A or 3A.

For example, $\cos^2 A=\frac{1}{2}(1+\cos 2A)$

and $\sin^3 A=\frac{1}{4}(3\sin A-\sin 3A)$

Exercise 15.

If $A+B+C=180^\circ$, prove that

1. $\sin 2A-\sin 2B+\sin 2C=4\cos A\sin B\cos C$ (J. & K. U. 1953)
2. $\cos 2A+\cos 2B+\cos 2C=-1-4\cos A\cos B\cos C$ (D. U. 1951)

   [**Hint** :—To get the common factor cos C, put $\cos 2C=2\cos^2 C-1$]
3. $\cos 2A+\cos 2B-\cos 2C=1-4\sin A\sin B\cos C$ (J. & K. U. 1949)
4. $\sin A+\sin B+\sin C=4\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}$ (P. U. 1948)

5. $\sin A+\sin B-\sin C=4\sin\frac{A}{2}\sin\frac{B}{2}\cos\frac{C}{2}$ (M. U.)

6. $\cos A+\cos B+\cos C=1+4\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$ (J & K. U. 1952)

7. $\tan A+\tan B+\tan C+\tan C=\tan A\tan B\tan C$ (J. & K. U. 1950)

8. $\cot A\cot B+\cot B\cot C+\cot C\cot A=1$

9. $\tan\frac{A}{2}\tan\frac{B}{2}+\tan\frac{B}{2}\tan\frac{C}{2}+\tan\frac{C}{2}\tan\frac{A}{2}=1$ (J. & K. U. 1961)

10. $\cos^2A+\cos^2B+\cos^2C=1-\cos A\cos B\cos C$ (J & K. U. 1954)

11. $\sin^2A+\sin^2B+\sin^2C=2+2\cos A\cos B\cos C$ (J. & K. U. 1960)

12. $\sin^2A+\sin^2B-\sin^2C=2\sin A\sin B\cos C$

13. $\cos^2A+\cos^2B-\cos^2C+2\sin A\sin B\cos C=1$

(**Hint.** Transpose $2\sin A\sin B\cos C$ to R. H. S. and proceed).

14. $\sin^2\frac{A}{2}+\sin^2\frac{B}{2}+\sin^2\frac{C}{2}=1-2\sin\frac{A}{2}\cos\frac{B}{2}\sin\frac{C}{2}$ (J. & K. U. 1959)

15. $\sin^2\frac{A}{2}+\sin^2\frac{B}{2}-\sin^2\frac{C}{2}=1-2\cos\frac{A}{2}\cos\frac{B}{2}\sin\frac{C}{2}$ (M. U. 1941)

16 $\sin 3A+\sin 3B+\sin 3C=-4\cos\frac{3A}{2}\cos\frac{3B}{2}\cos\frac{3C}{2}$

17. $\sin\frac{A}{2}+\sin\frac{B}{2}+\sin\frac{C}{2}=1+4\sin\frac{\pi-A}{4}\sin\frac{\pi-B}{4}\sin\frac{\pi-C}{4}$ (D.U.)

18. $\cos\frac{A}{2}+\cos\frac{B}{2}+\cos\frac{C}{2}=4\cos\frac{\pi-A}{4}\cos\frac{\pi-B}{4}$

$\cos\frac{\pi-C}{4}$ (D. U. 1935)

19. $\cos^2\frac{A}{2}+\cos^2\frac{B}{2}+\cos^2\frac{C}{2}$

$=2\left(1+\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}\right)$

20. $\sin(B+C-A)+\sin(C+A-B)+\sin(A+B-C)$
$=4\sin A\sin B\sin C$ (A. U. 1940)

21. $\cos\frac{A}{2}+\cos\frac{B}{2}+\cos\frac{C}{2}=4\cos\frac{B+C}{4}\cos\frac{C+A}{4}.$

$\cos\frac{A+B}{4}$ (A. U. 1939)

[**Hint.** Follows From Q. 18]

22. $\cos^3A+\cos^3B+\cos^3C=1+3\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$

$-\sin\frac{3A}{2}\sin\frac{3B}{2}\sin\frac{3C}{2}$

If $A+B+C+D=2\pi$, show that

23. $\cos A+\cos B+\cos C+\cos D=4\cos\frac{B+C}{2}.$

$\cos\frac{C+A}{2}\cos\frac{A+B}{}$,

24. $\sin A-\sin B+\sin C-\sin D=-4\cos\frac{A+B}{2}.$

$\sin\frac{A+C}{2}\cos\frac{A+D}{2}$

If $A+B+C=\pi$, show that

25. $\frac{\sin 2A+\sin 2B+\sin 2C}{\sin A+\sin B+\sin C}=8\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$

(J. & K. U. 1958)

26. $\dfrac{\text{Cos A}}{\text{Sin B sin C}}+\dfrac{\text{Cos B}}{\text{Sin C. Sin A}}+\dfrac{\text{Cos C}}{\text{Sin A. Sin B}}=2$

27. $\dfrac{\text{Cot A}+\text{Cot B}}{\tan A+\tan B}+\dfrac{\text{Cot B}+\text{Cot C}}{\tan B+\tan C}+\dfrac{\text{Cot C}+\text{Cot A}}{\tan C+\tan A}=1$

28. $\tan 3A+\tan 3B+\tan 3C=\tan 3A \tan 3B \tan 3C$.

Trigonometrical equations.

4. Let us consider the equation $\sin \theta=\frac{1}{2}$. The values of θ satisfying the equation are $30°, 150°, 390°,$......(when +ve angles are taken); and $-210°, -330°,$......(when negative angles are taken) Hence θ is *many valued*. Of all these values the numerically smallest value is called the *principal value* of θ. Therefore $30°$ is the principal value of angle θ satisying the equation $\sin 30°=\frac{1}{2}$. Given the principal value, we shall try to get a general expression (*i. e.* formula) for θ which includes all solutions (*i. e.* values) satisfying the equation.

Note. When n stands for an integer, an *even number* is algebraically expressed by $2n$ and an *odd number* by $(2n+1)$ or $2n-1$.

Also $(-1)^n$ is positive and $=1$, when n is even. It is negative and $=-1$, when n is odd

5. To find the general expression for all angles whose sine is zero.

Here it is required to solve the equation sine $\theta=0$.

If the sine of an angle θ is zero the revolving line must coincide with OX or OX′. Hence $\sin \theta=0$ for the values of θ given by $0, \pm\pi; \pm 2\pi, \pm 3\pi,$..... and so on.

The general expression $\theta=n\pi$ includes all the e values, where n is a positive integer or zero.

Hence when $\sin \theta=0$.

$\theta=n\pi$. where n is a positive or negative integer or **zero.**

6. To find the general expression for all angles whose cosine is zero.

Here we have to solve the equation $\cos \theta=0$.

If the cosine of an angle is Zero the revolving line must coincide with OY or OY′. Therefore the angle must be $\pm \frac{\pi}{2}$, or $\pm \frac{3\pi}{2}$, or $\pm \frac{5\pi}{2}$,...............*i. e.* the angles are odd multiples of $\frac{\pi}{2}$.

The general expression $\theta=(2n+1)\frac{\pi}{2}$ includes all these values, when n is a positive or negative integer or zero.

Hence if $\cos\theta=0$.

$\theta=(2\mathbf{n}+1)\frac{\pi}{2}$, where θ is a positive or negative integer or zero.

7. To find the general expression for all angles having a given sine.

Let α be the smallest positive or negative angle *in radians* having the given sine (=S, say) and θ any other angle having the same sine. Then, we have to find the general expression for all values of θ which satisfy the equation $\sin\theta=\sin\alpha$.

i. e. $\sin\theta-\sin\alpha=0$.

or $2\cos\frac{\theta+\alpha}{2}\sin\frac{\theta-\alpha}{2}=0$

$\therefore$ either $\cos\frac{\theta+\alpha}{2}=0$

$\therefore \frac{\theta+\alpha}{2}=(2r+1)\frac{\pi}{2}$ (Art. 5)

$\therefore \theta=(2r+1)\pi-\alpha$ ....(1)

$=$an odd multiple of $\pi-\alpha$

where r is zero, or any integer positive or negalive.

or $\sin\frac{\theta-\alpha}{2}=0$

or $\frac{\theta-\alpha}{2}=k\pi$ (Art 5)

or $\theta=2k\pi+\alpha$ ..... (2)

$=$an even multiple of $\pi+\alpha$

where k is zero, or any integer positive or negative.

The results (1) and (2) are included in the single formula.

$$\theta=\mathbf{n}\pi+(-1)^n\alpha \qquad ......(3)$$

where n is a positive or negative integer or zero.

[For when n is odd, Expression (3) agrees with [1) and when n is even (3) agrees with (2)].

Hence if $\sin\theta=\sin\alpha$, the general value of θ is given by :—

$\theta=\mathbf{n}\pi+(-\mathbf{1})^n\alpha$.

where n is any integer positive or negative or zero.

Cor. Since the **cosecant** is the reciprocal of the sine, if $\operatorname{cosec}\theta=\operatorname{cosec}\alpha$, then $\sin\theta=\sin\alpha$, so that the angles which have the same cosecant have the same sine.

Hence the general expression is $\theta=\mathbf{n}\pi+(-\mathbf{1})^n\alpha$

Note. α must be expressed in **radians** as all angles enter the formula in radians.

Ex. 1. Solve the equations :—

(*i*) $\sin\theta=\frac{\sqrt{3}}{2}$ (*ii*) $\sin\theta=-\frac{1}{\sqrt{2}}$

(*iii*) $\operatorname{cosec}\theta=2$.

Solution :—(*i*) $\sin\theta=\frac{\sqrt{3}}{2}$

The smallest (*i. e.*, Principal) value of θ satisfying the equation is 60° or $\frac{\pi^c}{3}$

$\therefore \sin\theta=\sin\frac{\pi}{3}$

$\therefore \theta=n\pi+(-1)^n.\frac{\pi}{3}$

(*ii*) $\sin\theta=-\frac{1}{\sqrt{2}}$

The smallest (*i. e.* Principal) value of θ satisfying the equation is -45° or $-\frac{\pi^c}{4}$

Hence $\sin\theta=\sin\left(-\frac{\pi}{4}\right)$

$$\therefore \theta = n\pi + (-1)^n \left(-\frac{\pi}{4}\right)$$

$$= n\pi - (-1)^n \frac{\pi}{4}.$$

(*iii*) $\operatorname{cosec} \theta = 2$

$\therefore \sin \theta = \frac{1}{2}$

The smallest (*i. e.* Principal) value of θ satisfying the equation is $30°$ *i e.* $\frac{\pi^c}{6}$

Hence the general value satisfying the equation is $\theta = n\pi + (-1)^n \frac{\pi}{6}$

8. To find the general expression for all angles having a given cosine.

Let α be the smallest positive or negative angle in *radians* having the given cosine $(=C$, say) and θ any other angle having the same cosine. Then, we have to find the general expression for all values of θ which satisfy the equation $\cos \theta = \cos \alpha$.

i. e. $\cos \theta - \cos \alpha = 0$

or $-2 \sin \frac{\theta+\alpha}{2} \sin \frac{\theta-\alpha}{2} = 0$

$\therefore$ either $\sin \frac{\theta+\alpha}{2} = 0$

$\therefore \frac{\theta+\alpha}{2} = r\pi$ (Art. 5)

$\therefore \theta = 2r\pi - \alpha$ ......(1)

$=$ an even multiple of π $-\alpha$

where r is any integer positive or negative or zero.

or $\sin \frac{\theta-\alpha}{2} = 0$

$\therefore \frac{\theta-\alpha}{2} = k\pi$ (Art. 5)

$\therefore \theta - \alpha = 2k\pi$

$\therefore \theta = 2k\pi + \alpha$ ......(2)

$=$ an even multiple of π $+\alpha$

where k is any integer positive or negative or zero.

Results (1) and (2) are both included in the formula $\theta = 2n\pi \pm \alpha$ where n is any positive or negative integer or zero.

Cor. Since secant is the reciprocal of the cosine, all angles which have the same secant, also have the same cosine and have, therefore, the same general expression $\theta=2n\pi\pm\alpha$.

9. To find the general expression for all angles having a given tangent.

Let α be the smallest positive or negative angle in radians having the given tangent ($=$T, say) and θ any other angle which has the same tangent. Then we have to find the general expression for all values of θ which satisfy the equation, $\tan\theta=\tan\alpha$.

i. e. $\tan\theta-\tan\alpha=0$

or $\dfrac{\sin\theta}{\cos\theta}-\dfrac{\sin\alpha}{\cos\alpha}=0$

or $\sin\theta\cos\alpha-\cos\theta\sin\alpha=0$

or $\sin(\theta-\alpha)=0$

$\therefore\ \theta-\alpha=n\pi$

$\therefore\ \theta=n\pi+\alpha$, where n is a positive or negative integer or zero.

Cor. Since the cotangent is the reciprocal of the tangent, all angles, having the same cotangent, have also the same tangent and are included in the general formula, $\theta=n\pi+\alpha$.

Ex. 1. Solve the equations,

$$(i)\ \cos\theta=\frac{1}{\sqrt{2}}\quad (ii)\ \sec\theta=-\frac{2}{\sqrt{3}}\quad (iii)\ \tan\theta=1$$

Solution : (*i*) $\cos\theta=\dfrac{1}{\sqrt{2}}$

The smallest value of θ satisfying the given equation is 45° or $\dfrac{\pi}{4}$.

$\therefore$ The general value of θ satisfying the given equation is $\theta=2n\pi\pm\dfrac{\pi}{4}$,

where n is any integer positive or negative or zero.

(*ii*) $\sec\theta = -\frac{2}{\sqrt{3}}$

$\therefore$ $\cos\theta = -\frac{\sqrt{3}}{2}$

$\therefore$ The smallest value of θ satisfying the equation is $150°$ or $\frac{5\pi}{6}$.

$\therefore$ The general value of θ is given by

$$\theta = 2n\pi \pm \frac{5\pi}{6},$$

where n is any integer positive or negative or zero.

(*iii*) $\tan\theta = 1$

The smallest value of θ satisfying the given equation is $\frac{\pi}{4}$.

$\therefore$ The general value of θ is given by

$$\theta = n\pi + \frac{\pi}{4},$$

where n is any integer positive or negative or zero.

Ex. 2. What is the general value of θ which satisfies both the equations

$$\sin\theta = -2, \ \cos\theta = \frac{-\sqrt{3}}{2}$$ (P. U. 1949)

Considering only the angles between $0°$ and $360°$. the values of θ satisfying $\sin\theta = -\frac{1}{2}$ are $210°$ and $330°$ Similarly the angles between $0°$ and $360°$ which

satisfy $\cos\theta = -\frac{\sqrt{3}}{2}$ are $150°$ and $210°$.

$\therefore$ The smallest value of θ which satisfies both the equations is $210°\left(i.e.\ \frac{7\pi^c}{6}\right)$

The most general value of θ will be obtained by adding any multiple of 2π to the angle.

$$\therefore \quad \theta=2n\pi+\frac{7\pi}{6}.$$

Ex. 3. Show that the general solution of the equation $\cos^2\theta=\cos^2\alpha$ is $\theta=n\pi\pm\alpha$.

First Method. $\cos^2\theta=\cos^2\alpha$

$$\begin{aligned}\therefore \quad \cos\theta &= \pm\cos\alpha \\ &= \cos\alpha \text{ and } \cos(\pi-\alpha) \\ &= (\text{an even multiple of } \pi)+\alpha \text{ or } (\pi-\alpha)\end{aligned}$$

$\therefore\ \theta=2n\pi\pm\alpha$

and $\theta=2n\pi\pm(\pi-\alpha)=(2n\pm1)\pi\pm\alpha$

$=$an odd multiple of $\pi\pm\alpha$

Both these are included in the formula $\theta=k\pi\pm\alpha$

Second Method. Multiplying both sides by 2, we have

$$2\cos^2\theta=2\cos^2\alpha$$

or $1+\cos 2\theta=1+\cos 2\alpha$

or $\cos 2\theta=\cos 2\alpha$

$\therefore\ 2\theta=2n\pi\pm2\alpha$

or $\theta=n\pi\pm\alpha.$

Note. We can show in a similar manner that the general solution of $\sin^2\theta=\sin^2\alpha$ and $\tan^2\theta=\tan^2\alpha$ is also given by $\theta=n\pi\pm\alpha$

Ex. 4. Solve the equation $2\sin^2\theta+\cos\theta=1$

Putting $\sin^2\theta=1-\cos^2\theta$, so that the equation has only one unknown T-ratio (*i. e.* $\cos\theta$) we get

$$2(1-\cos^2\theta)+\cos\theta=1$$

or $2\cos^2\theta-\cos\theta-1=0$

or $\cos\theta=\dfrac{1\pm\sqrt{1+8}}{4}=\dfrac{1\pm3}{4}=1$ and $-\frac{1}{2}$

(*i*) When $\cos\theta=1$

The least angle satisfying the equation is zero.

$\therefore\ \theta=n\pi$

(*ii*) When $\cos\theta = -\frac{1}{2}$

The numerically smallest angle satisying the equation is $120^\circ \left(i.\ e. \frac{2\pi}{3}\right)$

$\therefore\ \theta = 2n\pi \pm \frac{2\pi}{3}$.

Ex. 5. Solve $2\cos^2\theta - 3\cos\theta - 2 = 0$

Solving the quadratic, we have $\cos\theta = -\frac{1}{2}$ and 2.

But the 2nd solution is impossible as $\cos\theta$ is never > 1.

Hence the only admissible value is $\cos\theta = -\frac{1}{2}$ which gives

$$\theta = 2n\pi \pm \frac{2\pi}{3}.$$

Exercise 16.

1. Find the most general value of θ satisfying the equations :—

(*i*) $\sin\theta = \frac{\sqrt{3}}{2}$. (*ii*) $\cos\theta = \frac{1}{\sqrt{2}}$. (*iii*) $\tan\theta = \sqrt{3}$

(*iv*) $\cos\theta = -\frac{1}{2}$. (*v*) $\sec\theta = \frac{2}{\sqrt{3}}$. (*vi*) $\operatorname{cosec}\theta = \sqrt{2}$.

(*vii*) $\sin^2\theta = \frac{3}{4}$. (*viii*) $\cos^2\theta = \frac{1}{4}$. (*ix*) $3\tan^2\theta = 1$.

(P. U. 1943)

2. Find the most general solution of the simultaneous equations :—

(*i*) $\cos\theta = \frac{1}{\sqrt{2}}$, $\tan\theta = 1$ (*ii*) $\sin\theta = -\frac{1}{2}$, $\tan\theta = \frac{1}{\sqrt{3}}$

(P. U. 1942)

(*iii*) $\sin\theta = \frac{1}{2}$, $\cos\theta = -\frac{\sqrt{3}}{2}$

Solve the equations

3. $2\cos^2\theta - \sin\theta = 1$ 4. $4\sin^2\theta - 3\cos\theta = \frac{3}{2}$

5. $2\sin^2\theta+\sqrt{3}\cos\theta+1=0$ 6. $\tan^2\theta+\sec\theta-1=0$

7. $\sin^2\theta-2\cos\theta+\frac{1}{4}=0$ (P. U. 1951)

8. $\cos^2\theta-\sin\theta-\frac{1}{4}=0$ (J. & K U. 1954)

9. $2\cot^2\theta=\operatorname{cosec}^2\theta$

10. $3\tan\theta+\cot\theta=5\operatorname{cosec}\theta$ (J. & K. U. 1959)

[**Hint.** Change the T-ratios into sine and cosine].

11. $3\cos^2\theta-2\sqrt{3}\sin\theta\cos\theta-3\sin^2\theta=0$ (D. U. 1934)

[**Hint**. Divide by $\cos^2\theta$ and solve the quadratic equation in $\tan\theta$ thus obtained].

12. $5\tan^4\theta-1=4\tan^2\theta$ (P. U. 1953)

13. $a\cos^2\theta+b\sin^2\theta=c$ (J. & K. U. 1949)

14. $\tan^2\theta+\cot^2\theta=2$ (J. & K. U. 1955)

15. Solve $\left.\begin{array}{l}\cos(2x+3y)=\frac{1}{2}\\ \text{and } \cos(3x+3y)=\frac{\sqrt{3}}{2}\end{array}\right\}$ (P. U. 1944)

10. *Various other methods of solving equations of certain other types are best illustrated by examples.*

(*a*) To solve an equation of the type $a\cos\theta+b\sin\theta=c$

Here we first change L. H. S. into a single sine or cosine by art. 5 Chapter VII and then solve the equation.

Ex. 1. Solve the equation $\cos\theta+\sqrt{3}\sin\theta=\sqrt{2}$

First Method. Put $\left.\begin{array}{l}1=r\cos\alpha\\ \text{and } \sqrt{3}=r\sin\alpha\end{array}\right\}$, where r is positive.

Squaring and adding, $r=2$

Now $\cos\alpha=\frac{1}{r}=\frac{1}{2}$, $\sin\alpha=\frac{\sqrt{3}}{r}=\frac{\sqrt{3}}{2}$

$$\therefore \alpha=\frac{\pi}{3}.$$

The equation becomes $r(\cos\theta\cos\alpha+\sin\theta\sin\alpha)=\sqrt{2}$

$$\text{or}\quad 2\cos\left(\theta-\frac{\pi}{3}\right)=\sqrt{2}$$

$$\text{or} \quad \cos\left(\theta-\frac{\pi}{3}\right)=\frac{1}{\sqrt{2}}=\cos\frac{\pi}{4}$$

$$\therefore\ \theta-\frac{\pi}{3}=2n\pi\pm\frac{\pi}{4}$$

$$\text{or} \quad \theta=2n\pi\pm\frac{\pi}{4}+\frac{\pi}{3}$$

$$=2n\pi+\frac{7\pi}{12} \text{ and } 2n\pi+\frac{\pi}{12}.$$

Second Method. Putting $\left.\begin{array}{l} 1=r\sin\alpha \\ \text{and } \sqrt{3}=r\cos\alpha \end{array}\right\}$ so that $r=2$

and $\sin\alpha=\frac{1}{2}$, and $\cos\alpha=\frac{\sqrt{3}}{2}$, $\therefore\ \alpha=\frac{\pi}{6}$

Now we get, $r(\sin\alpha\cos\theta+\cos\alpha\sin\theta)=\sqrt{2}$
or $2\sin(\theta+\alpha)=\sqrt{2}$

$$\text{or } \sin\left(\theta+\frac{\pi}{6}\right)=\frac{1}{\sqrt{2}}=\sin\frac{\pi}{4}$$

$$\text{or } \theta+\frac{\pi}{6}=n\pi+(-1)^n\frac{\pi}{4}$$

$$\therefore\ \theta=n\pi+(-1)^n\frac{\pi}{4}-\frac{\pi}{6}.$$

Note. 1. This answer appears to be different from the first, but this is not so. For when n is odd, say $2r+1$

$$\theta=(2r+1)\pi-\frac{\pi}{4}-\frac{\pi}{6}=2r\pi+\pi-\frac{\pi}{4}-\frac{\pi}{6}=2r\pi+\frac{7\pi}{12} \qquad \ldots\ldots(1)$$

When n is even, say $2r$

$$\theta=2r\pi+\frac{\pi}{4}-\frac{\pi}{6}=2r\pi+\frac{\pi}{12} \qquad \ldots\ldots(2)$$

Thus the answers in the 2nd method, though apparently different, are really the same as obtained by the first method.

2. Squaring both sides of an equation should always be avoided since it gives, sometimes, extraneous solutions

which do not satisfy the given equation. For example, if the above equation is written as $(\sqrt{3}\sin\theta)^2=(\sqrt{2}-\cos\theta)^2$ and then solved, it will also give solution of the equation $-\sqrt{3}\sin\theta=\sqrt{2}-\cos\theta$ which is different from the given equation.

3. We notice that the first method gives a simpler result and is, therefore, to be preferred.

11. *Equations involving two or more multiple angles.*

Ex. 1. Solve the equation $\sin n\theta=\cos m\theta$. (P. U. 1942)

$$\sin n\theta=\sin\left(\frac{\pi}{2}-m\theta\right)$$

$\therefore\ n\theta=r\pi+(-1)^r\left(\frac{\pi}{2}-m\theta\right)$, where r is any positive or negative integer or zero. ......(A)

$$\therefore\ n\theta+(-1)^r m\theta=r\pi+(-1)^r\frac{\pi}{2}$$

$$\therefore\ \theta=\frac{r\pi+(-1)^r\frac{\pi}{2}}{n+(-1)^r m}.$$

Note. To avoid confusion we have used r instead of n as n already occurs in the equation.

Another Method. $\cos\left(\frac{\pi}{2}-n\theta\right)=\cos m\theta$

$$\therefore\ \frac{\pi}{2}-n\theta=2r\pi\pm m\theta$$

$$\therefore\ \theta=\frac{\frac{\pi}{2}-2r\pi}{n\pm m}$$

Note. From the above two examples we observe that the equation may be solved by any suitable method. The form of the answer does not matter. For, the answers though apparently different can be shown to be identical as in example 1.

Ex. 3. Solve the equation $\sin \theta + \sin 5\theta = \sin 3\theta$

(J. & K. U. 1961)

Since $\sin 5\theta + \sin \theta = 2 \sin 3\theta \cos 2\theta$.

$\therefore$ Substituting this value in the equation, we get

$2 \sin 3\theta \cos 2\theta = \sin 3\theta$, or $2 \sin 3\theta \cos 2\theta - \sin 3\theta = 0$

or $\sin 3\theta(2 \cos 2\theta - 1) = 0$

$\therefore$ either $\sin 3\theta = 0$

$\therefore$ $3\theta = n\pi + (-1)^n . (0)$

$\therefore$ $\theta = \dfrac{n\pi}{3}$

or $2 \cos 2\theta - 1 = 0$

$\therefore$ $\cos 2\theta = \frac{1}{2} = \cos \dfrac{\pi}{3}$

$\therefore$ $2\theta = 2n\pi \pm \dfrac{\pi}{3}$,

$\therefore$ $\theta = n\pi \pm \dfrac{\pi}{6}$

Exercise 17.

Solve the equations :—

1. $\sqrt{3} \cos \theta + \sin \theta = \sqrt{2}$ (C. U. 1938)
2. $\cos \theta + \sqrt{3} \sin \theta = 2$ (C. U. 1936)
3. $\sin \theta + \cos \theta = 1$ (J. & K. U. 1951)
4. $\cos \theta + \sin \theta = \sqrt{2}$ (J. & K. U. 1952)
5. $\sin \theta - \cos \theta = \dfrac{1}{\sqrt{2}}$ (J. & K. U. 1953)
6. $\sqrt{2} \sec \theta + \tan \theta = 1$ (P. U.)
   (**Hint**. Change into sine and cosine).
7. $\operatorname{cosec} x = \cot x + \sqrt{3}$ (M. U.)
8. $\sin m\theta = \sin n\theta$
9. $\sin 9\theta = \sin \theta$ (A. U. 1946)
10. $\cos 9\theta = \sin \theta$
11. $\tan 5\theta = \cot 2\theta$ (P. U. 1943)
12. $\tan m\theta = \tan n\theta$
13. $\sin 4\theta - \sin 2\theta = \cos 3\theta$

14. $\sin 3\theta+\sin 2\theta+\sin\theta=0$ (J. & K. U. 1958)

15. $\cos\theta-\cos 2\theta=\sin 3\theta$ (P. U. 1935)

16. $\cos\theta+\cos 3\theta+\cos 5\theta+\cos 7\theta=0$ (P. U. 1937)

17. $\cos 3\theta+\cos 2\theta+\cos\theta=0$ (J. & K. U. 1951)

18. $\tan\theta+\tan 2\theta+\tan 3\theta=0$ (J. & K. U. 1950)

19. $\tan 2\theta\tan\theta=1$ (P. U. 1951)

20. $\tan\left(\frac{\pi}{2}\sin\theta\right)=\cot\left(\frac{\pi}{2}\cos\theta\right)$ (P. U. 1949)

[**Hint.** $\tan\left(\frac{\pi}{2}\sin\theta\right)=\tan\left(\frac{\pi}{2}-\frac{\pi}{2}\cos\theta\right)$ etc]

21. $\sin 5\theta\cos 3\theta=\sin 9\theta\cos 7\theta$

(**Hint.** Change products into sums).

22. $\cos 3\theta-8\cos^3\theta=0$ (P. U. 1937)

23. $2(\sin^4\theta+\text{cas}^4\theta)=1$ (P. U. 1938)

24. $\sin\theta+\sin 7\theta=\sin 4\theta$ (J. & K. U. 1960)

CHAPTER IX

(Relations between the sides and the angles of a triangle)

1. It is a common practice to denote the angles of a triangle ABC by the capital letters A, B, C and the sides opposite to these angles by a, b, c respectively.

2. **Sine Formulae**. *The sides of any triangle are proportional to the sines of the opposite angles or in other words:—*

$$\frac{a}{\sin A}=\frac{b}{\sin B}=\frac{c}{\sin C}.$$

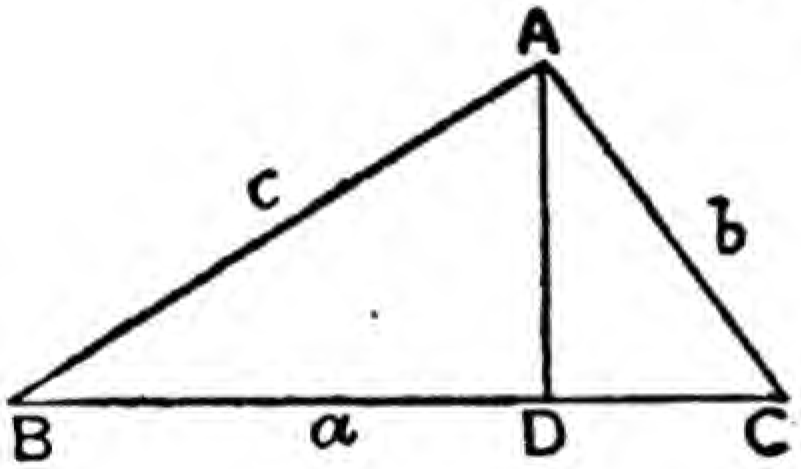


Fig. 1.

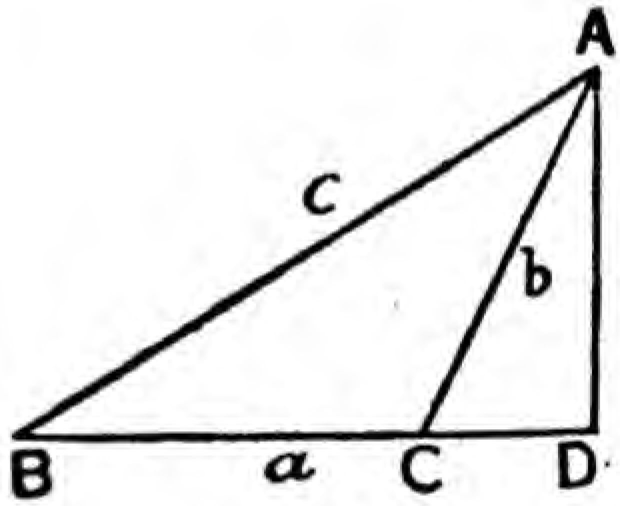


Fig. 2.

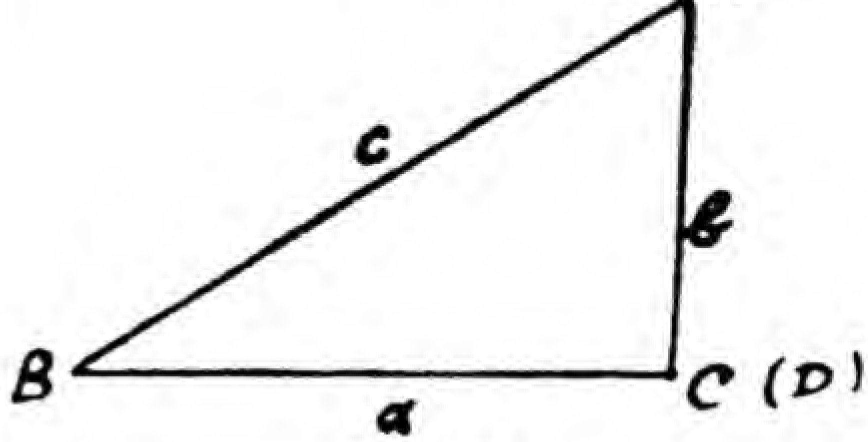


Fig. 3.

Let ABC be a triangle. One of the angles, say B, will be acute; C may then be acute, obtuse or a right angle. Draw AD$\perp$BC or BC produced.

Then, in all figs., from $\triangle$ ABC,

$$\frac{AD}{AB}=\sin B, \therefore AD=AB \sin B=c \sin B \quad \ldots\ldots\ldots(1)$$

Again, from rt. angled $\triangle$ ACD,

In Fig. 1, $\frac{AD}{AC}=\sin C$

In Fig. 2, $\frac{AD}{AC}=\sin ACD=\sin(\pi-C)=\sin C$

In Fig. 3, $\frac{AD}{AC}=1=\sin 90°=\sin C$

$\therefore$ In each case $AD=AC \sin C=b \sin C$ $\ldots\ldots\ldots(2)$

From (1) and (2), $b \sin C=c \sin B$

$$\text{or} \quad \frac{b}{\sin B}=\frac{c}{\sin C}$$

Similarly (By drawing $\perp$s on CA from B) we can prove

that $\frac{c}{\sin C}=\frac{a}{\sin A}$

$$\text{Hence } \frac{a}{\sin A}=\frac{b}{\sin B}=\frac{c}{\sin C}.$$

Cor. 1. If $a>b$, then $A>B$ *i. e.* the greater side has greater angle opposite to it.

Cor. 2. If $a=b$, then $A=B$ *i. e.* the angles opposite to equal sides are also equal.

Note. The above formulae are also called **The Law of Sines.**

Ex. 1. In any $\triangle$ ABC, prove that

$$\sin\frac{A-B}{2}=\frac{a-b}{c}\cos\frac{C}{2}$$

In the Sine Formula, let $\frac{a}{\sin A}=\frac{b}{\sin B}=\frac{c}{\sin C}=k$ (suppose)

$\therefore$ $a=k \sin A$, $b=k \sin B$, $c=k \sin C$

$$\text{Now } \frac{a-b}{c}=\frac{k\,(\sin A-\sin B)}{k\sin C}=\frac{\sin A-\sin B}{\sin C}$$

$$=\frac{2\cos\dfrac{A+B}{2}\sin\dfrac{A-B}{2}}{2\sin\dfrac{C}{2}\cos\dfrac{C}{2}}$$

$$=\frac{\cos\left(90^\circ-\dfrac{C}{2}\right)\sin\dfrac{A-B}{2}}{\sin\dfrac{C}{2}\cos\dfrac{C}{2}}$$

$$=\frac{\sin\dfrac{C}{2}\sin\dfrac{A-B}{2}}{\sin\dfrac{C}{2}\cos\dfrac{C}{2}}=\frac{\sin\dfrac{A-B}{2}}{\cos\dfrac{C}{2}}$$

$$\text{Cross multiplying, } \sin\frac{A-B}{2}=\frac{a-b}{c}\cos\frac{C}{2}$$

Observation :—We have started with the expression containing the sides on the R. H. S. and expressed it in terms of T-ratios of angles with the help of the sine formula.

Ex. 2. Prove that $\dfrac{a\sin(B-C)}{b^2-c^2}=\dfrac{b\sin(C-A)}{c^2-a^2}$

(Rajputana U.)

$$\text{Let } \frac{a}{\sin A}=\frac{b}{\sin B}=\frac{c}{\sin C}=k$$

$\therefore\ a=k\sin A,\ b=k\sin B,\ c=k\sin C$

$$\text{L. H. S.}=\frac{k\sin A\sin(B-C)}{k^2(\sin^2B-\sin^2C)}=\frac{\sin A\sin(B-C)}{k\sin(B+C)\sin(B-C)}$$

$$=\frac{\sin A}{k\sin(180^\circ-A)}=\frac{1}{k}$$

$$\text{R. H. S.}=\frac{k\sin B\sin(C-A)}{k^2(\sin^2C-\sin^2A)}=\frac{\sin B\sin(C-A)}{k\sin(C+A)\sin(C-A)}$$

$$=\frac{\sin B}{k\sin(180°-B)}=\frac{1}{k}$$

$\therefore$ L. S. = R. H. S.

3. Napier's Analogy. To prove that in any $\triangle$ ABC,

$$\tan\frac{A-B}{2}=\frac{a-b}{a+b}\cot\frac{C}{2}.$$ (P. U. 1953)

From the Sine Formula $\frac{a}{\sin A}=\frac{b}{\sin B}$, we have,

$$\frac{a}{b}=\frac{\sin A}{\sin B}$$

$\therefore$ by componendo and dividendo,

$$\frac{a-b}{a+b}=\frac{\sin A-\sin B}{\sin A+\sin B}$$

$$=\frac{2\cos\frac{A+B}{2}\sin\frac{A-B}{2}}{2\sin\frac{A+B}{2}\cos\frac{A-B}{2}}=\frac{\tan\frac{A-B}{2}}{\tan\frac{A+B}{2}}$$

$$=\frac{\tan\frac{A-B}{2}}{\tan\left(90°-\frac{C}{2}\right)}=\frac{\tan\frac{A-B}{2}}{\cos\frac{C}{2}}$$

$\therefore$ Cross-multiplying, $\tan\frac{A-B}{2}=\frac{a-b}{a+b}\cot\frac{C}{2}$

Similarly we can show that

$$\tan\frac{B-C}{2}=\frac{b-c}{b+c}\cot\frac{A}{2}$$ ...... (J. & K. 1961)

and $\tan\frac{C-A}{2}=\frac{c-a}{c+a}\cot\frac{B}{2}$

Note. This result is also called the **Tangent Formula.** It was stated by Napier in the form of the

$$\text{proportion,}\quad \frac{\tan \frac{A-B}{2}}{\tan \frac{A+B}{2}}=\frac{a-b}{a+b}$$

Exercise 18.

In any $\triangle$ ABC, prove that

1. $\cos\frac{B-C}{2}=\frac{b+c}{a}\sin\frac{A}{2}$ (J. & K. U. 1954)

2. $\sin\frac{B-C}{2}=\frac{b-c}{a}\cos\frac{A}{2}$ (M. U. 1949)

3. $a\sin(B-C)+b\sin(C-A)+c\sin(A-B)=0$ (P. U. 1950)

4. $\frac{\sin(B-C)}{\sin(B+C)}=\frac{b^2-c^2}{a^2}$

5. $\frac{\tan\frac{B-C}{2}}{\tan\frac{B+C}{2}}=\frac{b-c}{b+c}$ (J. & K. U. 1957)

6. $a\sin A-b\sin B=c\sin(A-B)$

7. $a(\cos B+\cos C)=2(b+c)\sin^2\frac{A}{2}$

8. $a\sin\left(\frac{A}{2}+B\right)=(b+c)\sin\frac{A}{2}$ (P. U. 1943)

9. $\frac{b\sec B+c\sec C}{\tan B+\tan C}=\frac{c\sec C+a\sec A}{\tan C+\tan A}=\frac{a\sec A+b\sec B}{\tan A+\tan B}$ (D. U. 1944)

10. $\cos\frac{B-C}{2}=2\sin\frac{A}{2}$, when $b+c=2a$ (J. & K. U. 1952)

11. $a\cos A+b\cos B+c\cos C=2a\sin B\sin C$
$=2b\sin C\sin A=2c\sin A\sin B.$

12. $\frac{a^2\sin(B-C)}{\sin B+\sin C}+\frac{b^2\sin(C-A)}{\sin C+\sin A}+\frac{c^2\sin(A-B)}{\sin A+\sin B}=0$

13. $\dfrac{a \sin (B-C)}{b^2-c^2}=\dfrac{b \sin (C-A)}{c^2-a^2}=\dfrac{c \sin (A-B)}{a^2-b^2}$

(J. & K. U. 1961)

14. $\dfrac{a+b+c}{a-b+c}=\cot \dfrac{A}{2} \cot \dfrac{C}{2}$.

4. The Cosine Formula.

To prove that in any $\triangle$ ABC,

$$\cos C=\frac{a^2+b^2-c^2}{2ab}$$

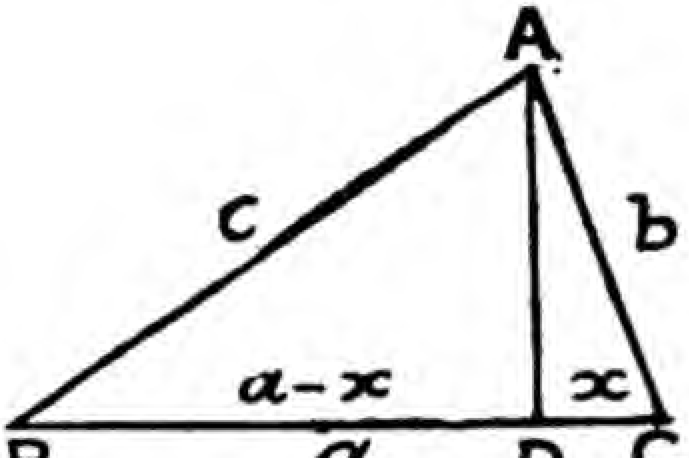


Fig. 1.

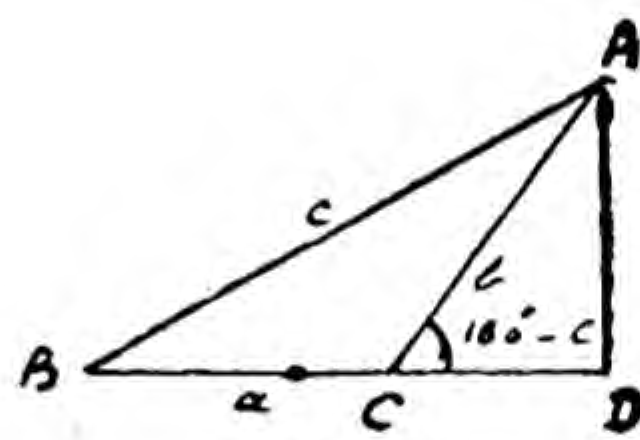


Fig. 2.

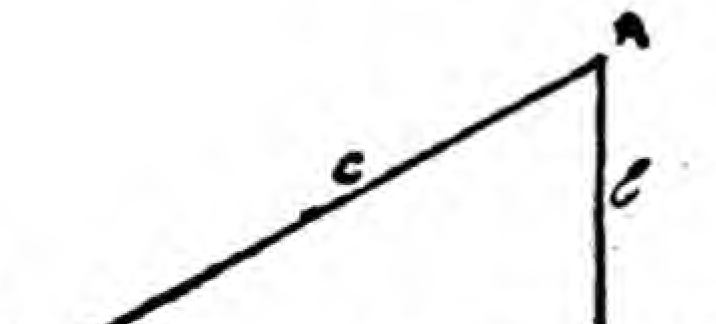


Fig. 3.

Let ABC be the triangle. One of its angles say B, will be acute; C may then be acute, obtuse or a right angle.

Draw AD ⊥ BC or BC produced. Let CD $=x$

From rt. angled $\triangle$ ABC,

In fig. 1, $c^2=BD^2+DA^2=(a-x)^2+DA^2$

$=a^2-2ax+x^2+DA^2=a^2-2ax+b^2$, [$\because x^2+DA^2=b^2$]

But $x=b\cos C$, $\therefore c^2=a^2+b^2-2ab\cos C$

In fig. 2, $c^2=BD^2+DA^2=(a+x)^2+DA^2$

$=a^2+2ax+x^2+DA^2$

$=a^2+2ax+b^2$

But $x=b\cos ACD=b\cos(180°-C)=-b\cos C$

$\therefore c^2=a^2+b^2-2ab\cos C$

In fig. 3, $c^2=a^2+b^2$

$=a^2+b^2-2ab\cos C$ [$\because$ cor $C=\cos 90°=0$]

Thus in all cases, $\mathbf{c^2=a^2+b^2-2ab\cos C}$

$\therefore 2ab\cos C=a^2+b^2-c^2$

$$\therefore \cos C=\frac{\mathbf{a^2+b^2-c^2}}{\mathbf{2ab}}.$$

Similarly $\mathbf{a^2=b^2+c^2-2bc}\cos\mathbf{A}$, **or cos** $\mathbf{A}=\dfrac{\mathbf{b^2+c^2-a^2}}{\mathbf{2bc}}$

and $\mathbf{b^2=c^2+a^2-2ca\cos B}$, **or cos** $\mathbf{B}=\dfrac{\mathbf{c^2+a^2-b^2}}{\mathbf{2ca}}$

Note. The above formulae enable us to express the cosine of an angle of a triangle in terms of the sides.

Ex 1. In any $\triangle$ ABC, prove that $a(b\cos C-c\cos B)=b^2-c^2$ (J. & K. U. 1951)

Putting values of cos B and cos C from the cosine formula we get

$$\text{L. H. S.}=a\left(b.\ \frac{a^2+b^2-c^2}{2ab}-c.\ \frac{c^2+a^2-b^2}{2ca}\right)$$

$$=\frac{a^2+b^2-c^2}{2}-\frac{c^2+a^2-b^2}{2}=\frac{a^2+b^2-c^2-c^2-a^2+b^2}{2}=b^2-c^2$$

Ex. 2. In any $\triangle$ ABC, prove that

$$(b^2-c^2)\cot A+(c^2-a^2)\cot B+(a^2-b^2)\cot C=0$$

(P. U. 1944)

Since $\cos A=\dfrac{b^2+c^2-a^2}{2bc}$ and $\dfrac{a}{\sin A}=k$ gives $\sin A=\dfrac{a}{k}$

$$\therefore \text{ 1st term on L. H. S.} = (b^2-c^2)\frac{\cos A}{\sin A}$$

$$=(b^2-c^2).\frac{b^2+c^2-a^2}{2bc}\cdot\frac{k}{a}$$

$$=\frac{k}{2abc}[(b^4-c^4)-a^2(b^2-c^2)]$$

Similarly, 2nd term$=\dfrac{k}{2abc}[(c^4-a^4)-b^2(c^2-a^2)]$

and 3rd term$=\dfrac{k}{2abc}[(a^4-b^4)-c^2(a^2-b^2)]$

$$\therefore \text{L.H.S.}=\frac{k}{2abc}[(b^4-c^4)+(c^4-a^4)+(a^4-b^4)$$
$$-a^2(b^2-c^2)-b^2(c^2-a^2)-c^2(a^2-b^2)]$$
$$=\frac{k}{2abc}[0]=0$$

5. The Projection Formulae.

To prove that in $\triangle$ ABC,

$$\mathbf{a=b\cos C+c\cos B,\ b=c\cos A+a\cos C}$$
$$\mathbf{and\ c=a\cos B+b\cos A.}$$

Let ABC be a triangle. One of its angles, say B, will be acute; C may then be acute, obtuse or a right angle. Draw AD$\perp$BC or BC produced. [See figs. of art. 4]

In fig. 1, BC=BD+DC ..... (1)

But $\dfrac{BD}{AB}=\cos B \therefore BD=AB\cos B=c\cos B$

and $\dfrac{CD}{AC}=\cos C \therefore CD=AC\cos C=b\cos C$

$\therefore$ from (1) $a=c\cos B+b\cos C$

In fig. 2, BC=BD$-$CD

$$=c\cos B-b\cos\angle ACD$$
$$=c\cos B-b\cos(180^\circ-C)$$
$$=c\cos B+b\cos C$$

In fig. 3, $BC = c \cos B$

$$= c \cos B + b \cos C \quad (\because \cos C = \cos 90° = 0)$$

Thus in all cases, $a = b \cos C + c \cos B$

Similarly $b = c \cos A + a \cos C$

and $c = a \cos B + b \cos A$

Ex. 3. Prove that in any $\triangle ABC$

$$(b+c) \cos A + (c+a) \cos B + (a+b) \cos C = a+b+c$$

(M. U. 1946)

Opening brackets and grouping terms, we get

$$\text{L. H. S.} = (a \cos B + b \cos A) + (b \cos C + c \cos B) + (c \cos A + a \cos C)$$

$$= c + a + b \qquad \text{(by the projection formulae)}$$

$$= a + b + c$$

Exercise 19.

In any $\triangle ABC$, prove that :—

1. $\dfrac{\cos A}{a} + \dfrac{\cos B}{b} + \dfrac{\cos C}{c} = \dfrac{a^2+b^2+c^2}{2abc}$

2. $bc \cos A + ca \cos B + ab \cos C = \frac{1}{2}(a^2+b^2+c^2)$

3. $c(b \cos A - a \cos B) = b^2 - a^2$

4. $b \cos B + c \cos C = a \cos (B - C)$

5. $b^2 \cos 2A - a^2 \cos 2B = b^2 - a^2$

6. $(a^2 - b^2 + c^2) \tan B = (a^2 + b^2 - c^2) \tan C$ (B.U.)

7. $\dfrac{\cos B}{\cos C} = \dfrac{c - b \cos A}{b - c \cos A}$

8. $\dfrac{b^2 - c^2}{a^2} \sin 2A + \dfrac{c^2 - a^2}{b^2} \sin 2B + \dfrac{a^2 - b^2}{c^2} \sin 2C = 0$

(A. U.)

9. $c(\cos A + \cos B) = 2(a+b) \sin^2 \dfrac{C}{2}$

10. (*i*) If $a \cos A = b \cos B$, then either the triangle is isosceles or right angled.

(*ii*) If $\frac{\cos A}{a} = \frac{\cos B}{b}$, then the triangle is isosceles. (C. U.)

11. $c^2 = (a-b)^2 \cos^2 \frac{C}{2} + (a+b)^2 \sin^2 \frac{C}{2}$

[**Hint.** Use the Formulae

$2 \sin^2 \frac{C}{2} = 1 - \cos C,\ 2 \cos^2 \frac{C}{2} = 1 + \cos C$]

12. If $C = 60°$, then $\frac{1}{a+c} + \frac{1}{b+c} = \frac{3}{a+b+c}$ (Pat. U. 1937)

13. If a, b, c are 3, 5, 7 respectively show that (*i*) the triangle has an obtuse angle equal to 120° and that (*ii*) the ratio into which the greatest side is divided by the perpendicular from the opposite angle is 33 : 65. (D. U.)

14. The sines of the angles of a triangle are 5 : 7 : 8, prove that the cosines of the angles are as 11 : 7 : 2.

Half-angle Formulae.

In the following articles s denotes semi-perimeter so that $s = \frac{a+b+c}{2}$.

6. To find the sines of half the angles of a triangle in terms of its sides.

i. e. to prove that in any △ ABC.

$$\sin \frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{bc}},$$

We know that $\cos A = \frac{b^2 + c^2 - a^2}{2bc}$

and $\cos A = 1 - 2 \sin^2 \frac{A}{2}$

$$\therefore\ 2\sin^2\frac{A}{2}=1-\cos A$$

$$=1-\frac{b^2+c^2-a^2}{2bc}\quad \text{(Cosine Formula)}$$

$$=\frac{2bc-b^2-c^2+a^2}{2bc}$$

$$=\frac{a^2-(b-c)^2}{2bc}=\frac{(a+b-c)\ (a-b+c)}{2bc} \quad \ldots\ldots (1)$$

Now $a+b+c=2s$

$$\therefore\ a+b-c=a+b+c-2c=2s-2c=2(s-c)$$

and $a-b+c=a+b+c-2b=2s-2b=2(s-b)$

$$\therefore \text{ from (1), } 2\sin^2\frac{A}{2}=\frac{2(s-c)\ .\ 2(s-b)}{2bc}$$

$$\therefore\ \sin^2\frac{A}{2}=\frac{(s-b)\ (s-b)}{bc}$$

$$\therefore\ \sin\frac{A}{2}=\sqrt{\frac{(s-b)\ (s-c)}{bc}}$$

Similarly $\sin\dfrac{B}{2}=\sqrt{\dfrac{(s-c)\ (s-a)}{ca}}$

$$\sin\frac{C}{2}=\sqrt{\frac{(s-a)\ (s-b)}{ab}}$$

Where positive sign is taken with the radicals because each of the angles A,B,C, is $<180^\circ$ and hence $\frac{A}{2}$, $\frac{B}{2}$, $\frac{C}{2}$ are each $<90^\circ$ and lie in the first quadrant.

7. To find the cosines of half the angles of a triangle in terms of its sides

i. e. To prove that in any $\triangle ABC$,

$$\cos\frac{A}{2}=\sqrt{\frac{s(s-a)}{bc}},\quad \cos\frac{B}{2}=\sqrt{\frac{s(s-b)}{ca}}$$

$$\text{and } \cos\frac{C}{2}=\sqrt{\frac{s(s-c)}{ab}}$$

We know that $\cos A = \dfrac{b^2+c^2-a^2}{2bc}$

and $\cos A = 2\cos^2\dfrac{A}{2} - 1$

$\therefore\ 2\cos^2\dfrac{A}{2} = 1+\cos A$

$$= 1 + \frac{b^2+c^2-a^2}{2bc}$$

$$= \frac{2bc+b^2+c^2-a^2}{2bc} = \frac{(b+c)^2-a^2}{2bc}$$

$$= \frac{(b+c+a)\ (b+c-a)}{2bc} \qquad \ldots\ldots(1)$$

Now $a+b+c=2s$

$\therefore\ b+c-a = b+c+a-2a = 2(s-a)$

$\therefore$ from (1), $2\cos^2\dfrac{A}{2} = \dfrac{2s\cdot 2(s-a)}{2bc}$

$$\therefore\ \cos^2\frac{A}{2} = \frac{s(s-a)}{bc}\,.$$

$$\therefore\ \cos\frac{A}{2} = \sqrt{\frac{s(s-a)}{bc}}$$

$$\text{Similarly}\ \cos\frac{B}{2} = \sqrt{\frac{s(s-b)}{ca}}$$

$$\text{and}\ \cos\frac{C}{2} = \sqrt{\frac{s(s-c)}{ab}}$$

Where positive sign is taken with the radicals because each of the angels A, B, C is $<180°$ and hence $\dfrac{A}{2}$, $\dfrac{B}{2}$, $\dfrac{C}{2}$ are each $<90°$ and so lie in the first quadrant.

8. To find the tangents of half the angles of a triangle in terms of its sides.

i. e. To prove that in any triangle ABC,

$$\tan\frac{A}{2}=\sqrt{\frac{(s-b)(s-c)}{s(s-a)}}. \qquad \text{(P. U. 1955)}$$

We know that $\cos A=\dfrac{b^2+c^2-a^2}{2bc}$

$$\text{and } \tan^2\frac{A}{2}=\frac{2\sin^2\frac{A}{2}}{2\cos^2\frac{A}{2}}=\frac{1-\cos A}{1+\cos A}$$

$$=\frac{1-\dfrac{b^2+c^2-a^2}{2bc}}{1+\dfrac{b^2+c^2-a^2}{2bc}}=\frac{a^2-(b-c)^2}{(b+c)^2-a^2}$$

$$=\frac{(a+b-c)(a-b+c)}{(b+c+a)(b+c-a)}=\frac{2(s-c).2(s-b)}{2s.\,2(s-a)}$$

$$=\frac{(s-b)(s-c)}{s(s-a)}$$

$$\therefore \tan\frac{A}{2}=\sqrt{\frac{(s-b)(s-c)}{s(s-a)}}$$

Or thus : $\tan\dfrac{A}{2}=\dfrac{\sin\frac{A}{2}}{\cos\frac{A}{2}}=\dfrac{\sqrt{\dfrac{(s-b)(s-c)}{bc}}}{\sqrt{\dfrac{s(s-a)}{bc}}}$

$$=\sqrt{\frac{(s-b)(s-c)}{s(s-a)}}.$$

Similarly $\tan\dfrac{B}{2}=\sqrt{\dfrac{(s-c)(s-a)}{s(s-b)}}$

and $\tan\dfrac{C}{2}=\sqrt{\dfrac{(s-a)(s-b)}{s(s-c)}}.$

9 To prove that in any triangle ABC,

$$\textbf{Sin A} = \frac{2}{bc}\sqrt{s(s-a)(s-b)s-c)}.$$

$$\sin A = 2\sin\frac{A}{2}\cos\frac{A}{2}$$

$$= 2\sqrt{\frac{(s-b)(s-c)}{bc}}\cdot\sqrt{\frac{s(s-a)}{bc}}$$

$$= \frac{2}{bc}\sqrt{s(s-a)(s-b)(s-c)}.$$

Similarly $\sin B = \dfrac{2}{ca}\sqrt{s(s-a)\,s-b)\,(s-c)}$

and $\sin C = \dfrac{2}{ab}\sqrt{s(s-a)(s-b)(s-c)}$

Cor. $\dfrac{\sin A}{a} = \dfrac{\sin B}{b} = \dfrac{\sin C}{c} = \dfrac{2\sqrt{s(s-a)(s-b)(s-c)}}{abc}$

This is, incidentally, a verification of the Sine formulae.

Ex. 1. In any $\triangle ABC$, prove that

$$c+a-b = 2\left(a\sin^2\frac{C}{2} + c\sin^2\frac{A}{2}\right)$$

$$\text{R.H.S.} = 2\left[a.\frac{(s-a)(s-b)}{ab} + c.\frac{(s-b)\,s-c)}{bc}\right]$$

$$= \frac{2(s-b)}{b}\left[(s-a) + (s-c)\right] = \frac{(2s-2b)}{b}[2s-a-c]$$

$$= \frac{(a-b+c)(b)}{b} = a+c-b = \text{L.H.S.}$$

Ex. 2. If $a, b, c,$ are in A. P. show that

$$\tan\frac{A}{2}\tan\frac{C}{2} = \frac{1}{3}.$$

The result is true, if

$$\sqrt{\frac{(s-b)(s-c)}{s(s-a)}}\cdot\sqrt{\frac{(s-a)(s-b)}{s(s-c)}} = \tfrac{1}{3}$$

[Putting values of tan $\frac{A}{2}$ and tan $\frac{C}{2}$]

or if $3s-3b=s$

or if $2s.=3b$

or if $a+b+c=3b$

or if $a+c=2b$

i. e. if a, b, c are in A. P., which is given.

Hence the given result is proved.

Ex. 8. If $\cot\frac{A}{2}=\frac{b+c}{a}$, show that the triangle is rt. angled. (P. U. 1937)

$$\cot\frac{A}{2}=\sqrt{\frac{s(s-a)}{(s-b)(s-c)}}$$

$$=\sqrt{\frac{(a+b+c)(b+c-a)}{(a-b+c)(a+b-c)}}$$

$$=\sqrt{\frac{(b+c)^2-a^2}{a^2-(b-c)^2}} \qquad \ldots\ldots(1)$$

But it is given that $\cot\frac{A}{2}=\frac{b+c}{a}$ ......(2)

$\therefore$ from (1) and (2) after squaring,

$$\frac{(b+c)^2-a^2}{a^2-(b-c)^2}=\frac{(b+c)^2}{a^2}$$

or $a^2(b+c)^2-a^4=a^2(b+c)^2-(b^2-c^2)^2$

or $a^4=(b^2-c^2)^2$

or $a^2=b^2-c^2$ (Taking sq. root).

$\therefore a^2+c^2=b^2$

Hence the triangle is right angled, $\angle$B being the right angle.

EXERCISE 20.

In any $\triangle ABC$, show that :—

1. $a\cos^2\frac{B}{2}+b\cos^2\frac{A}{2}=s.$

2. $(b+c-a)\sin\frac{A}{2}=2a\sin\frac{B}{2}\sin\frac{C}{2}.$

3. $\dfrac{\cot\frac{B}{2}}{\cot\frac{C}{2}}=\dfrac{a-b+c}{a+b-c}.$

4. $\frac{2(a+b)}{c}\sin^2\frac{C}{2}=\cos A+\cos B.$

5. $a(\cos B+\cos C)=2(b+c)\sin^2\frac{A}{2}.$

6. If $3a=b+c$, then $\cot\frac{B}{2}\cot\frac{C}{2}=2.$

7. $c\left(\tan\frac{A}{2}-\tan\frac{B}{2}\right)=(a-b)\left(\tan\frac{A}{2}+\tan\frac{B}{2}\right)$

8. If the sides of a triangle are in A. P. then the cotangents of half the angles are also in A P.

(P.U. 1943)

9 $\dfrac{\cos^2\frac{A}{2}}{a}+\dfrac{\cos^2\frac{B}{2}}{b}+\dfrac{\cos^2\frac{C}{2}}{c}=\dfrac{s^2}{abc}$ (M. U)

10. If $\cot A+\cot C=2\cot B$, then $c^2+a^2=2b^2$.

(A. U. 1943)

11. If a, b, c are in A. P., then $2\sin\frac{A}{2}\sin\frac{C}{2}=\sin\frac{B}{2}$

(P. U. 1949)

(12) If a, b, c are in H. P., prove that :

$$\sin^2 \frac{A}{2}, \sin^2 \frac{B}{2}, \sin^2 \frac{C}{2} \text{ are also in H. P.}$$

(J. & K. U. 1957)

(13) If $a \cos^2 \frac{C}{2} + c \cos^2 \frac{A}{2} = \frac{3b}{2}$, show that the sides a, b, c are in A. P.

(J. & K. U. 1959).

14. The bisector of the angle A of a $\triangle$ meets BC in D,

Show that $BD = \frac{a \sin C}{\sin C + \sin B}$, $DC = \frac{a \sin B}{\sin C + \sin B}$

$$\text{and } AD = \frac{2bc}{b+c} \cos \frac{A}{2}.$$

15. Show that a triangle having sides equal to 3, 5, 7 is an obtuse-angled triangle and determine the obtuse angle.

[**Hint.** Use cosine formula]

16 If $\cot \frac{A}{2}$, $\cot \frac{B}{2}$, $\cot \frac{C}{2}$ be in A. P.,

then $\cot \frac{A}{2} \cdot \cot \frac{C}{2} = 3$.

17. If $3 \tan \frac{A}{2} \tan \frac{C}{2} = 1$, prove that a,b,c, are in A.P.

10. To find the area of a triangle ABC.

(a) *Area of the triangle when two sides and the included angle are given.*

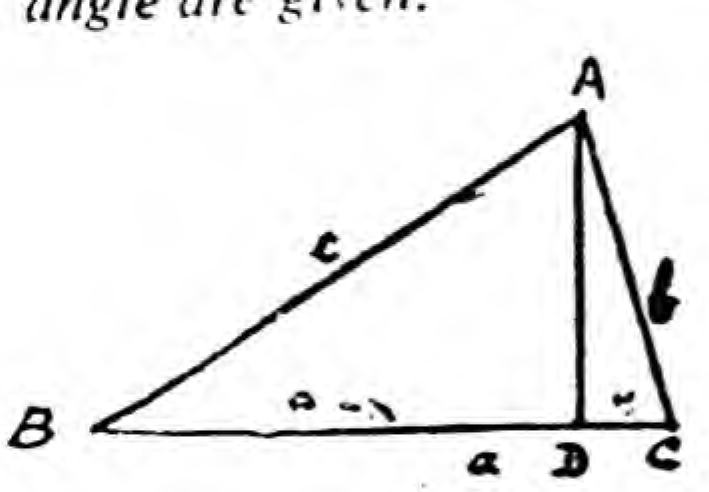


Fig. 1.

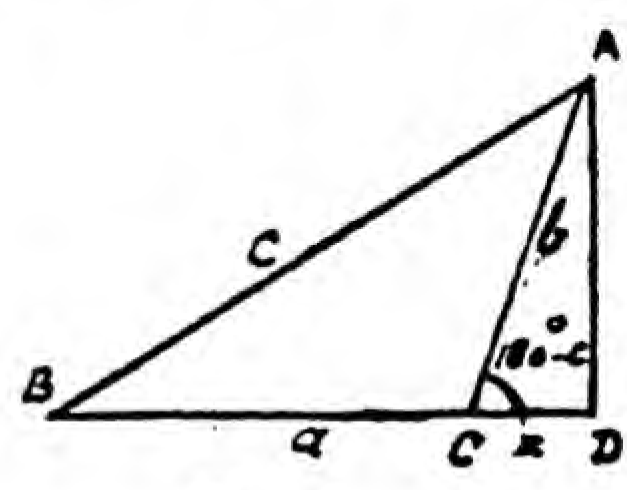


Fig. 2.

Let $\triangle$ denote the area of the triangle.

Draw $AD \perp BC$ or BC produced.

Then $\triangle = \frac{1}{2}$ BC . AD ..... (1)

But from $\triangle$ ACD

In fig. 1, $AD = b \sin C$

In fig. 2, $AD = b \sin (180° - C) = b \sin C$

$\therefore$ From (1), $\triangle = \frac{1}{2} a \cdot b \sin C = \frac{1}{2} ab \sin C$

Similarly, we can prove that $\triangle = \frac{1}{2} bc \sin A$

$= \frac{1}{2} ca \sin B.$

Rule :—Area of a triangle $= \frac{1}{2}$ (product of two sides)

$\times$ (sine of the included angle).

(*b*) *Area of a triangle when 3 sides a, b, c are given*

We know that $\sin A = \dfrac{2}{bc} \sqrt{s(s-a)(s-b)(s-c)}$

(Art 9)

$$\therefore \triangle ABC = \tfrac{1}{2} bc \sin A = \tfrac{1}{2} bc \cdot \frac{2}{bc} \sqrt{s(s-a)(s-b)(s-c)}.$$

$$= \sqrt{(s-a)(s-b)(s-c)}.$$

This is known as **Hero's formula.** (1)

(*c*) *Area in terms of one side and two angles* (P. U. 1936)

$$\text{Since} \frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}.$$

$$\therefore a = \frac{c \sin A}{\sin C}, \text{and } b = \frac{c \sin B}{\sin C}$$

$$\therefore \triangle ABC = \tfrac{1}{2} ab \sin C = \tfrac{1}{2} \frac{c \sin A}{\sin C} \cdot \frac{c \sin B}{\sin C} \cdot \sin C$$

$$= \tfrac{1}{2} \frac{c^2 \sin A \sin B}{\sin C}$$

$$=\frac{1}{2}\frac{c^2 \sin A \sin B}{\sin (A+B)} \quad [\because C=180^\circ-(A+B)]$$

Similarly $\triangle=\frac{1}{2}\frac{a^2 \sin B \sin C}{\sin (B+C)}$

$$\text{or}=\frac{1}{2}\frac{b^2 \sin C \sin A}{\sin (C+A)}$$

Cor. $\triangle=\frac{1}{2}\frac{a^2 \sin B \sin C}{\sin A}=\frac{1}{2}\frac{b^2 \sin C \sin A}{\sin B}$

$$=\frac{1}{2}\frac{c^2 \sin A \sin B}{\sin C}$$

Ex. 1. Find the area of the triangle whose sides are 5 ft, 6 ft. 7 ft.

Here $s=\frac{1}{2}(5+6+7)=9$

$s-a=9-5=4$

$s-b=9-6=3$

$s-c=9-7=2$

$\therefore \triangle=\sqrt{s(s-a)(s-b)(s-c)}=\sqrt{9\times4\times3\times2}=6\sqrt{6}$ sq ft.

Exercise 21.

Find the area of the $\triangle$ ABC, when

(1) $a=5$ ft., $b=7$ ft., $c=8$ ft.

(2) $a=13$ ft., $b=14$ ft., $c=15$ ft.

(3) $a=18$, $b=24$, $C=30^\circ$.

(4) $b=8$ ft., $c=9$ ft, $A=3^\circ$.

(5) $a=12$ ft., $B=60^\circ$, $C=45^\circ$.

CHAPTER X

Radii of the circles connected with a triangle.

1. Definitions.

Circumcircle :—The circle which passess through the vertices of a triangle is called the circumcircle of the triangle. Its centre, called the *circumcentre*, is the point of intersection of the right bisectors of the sides of the triangle. Its radius is known as *circum-radius* and is denoted by R.

Incircle : The circle which touches the sides of a triangle internally is called the incircle. Its centre, called the *in-centre, is* the point of intersection of the internal bisectors of the angles of the triangle and is denoted by I. Its radius, known as *inradius*, is denoted by *r*.

Escribed circle : The circle which touches the side BC of the triangle ABC and the other two sides AB, BC produced is called the *escribed circle* opposite to the vertex A. Its centre called the *ex-centre*, is the point of intersection of the external bisectors of the angles B, C and the internal bisector of angle A and is denoted by I_1. Its radius, known as *ex-radius* is denoted by r_1. Similarly the centres of escribed circles opposite to the vertices B and C are denoted by I_2, I_3 and the corresponding ex-radii by r_2, r_3 respectively.

2. To prove that in any triangle ABC,

(*a*) $R = \dfrac{a}{2\sin A} = \dfrac{b}{2\sin B} = \dfrac{c}{2\sin C}$ (J. & K. U. 1959)

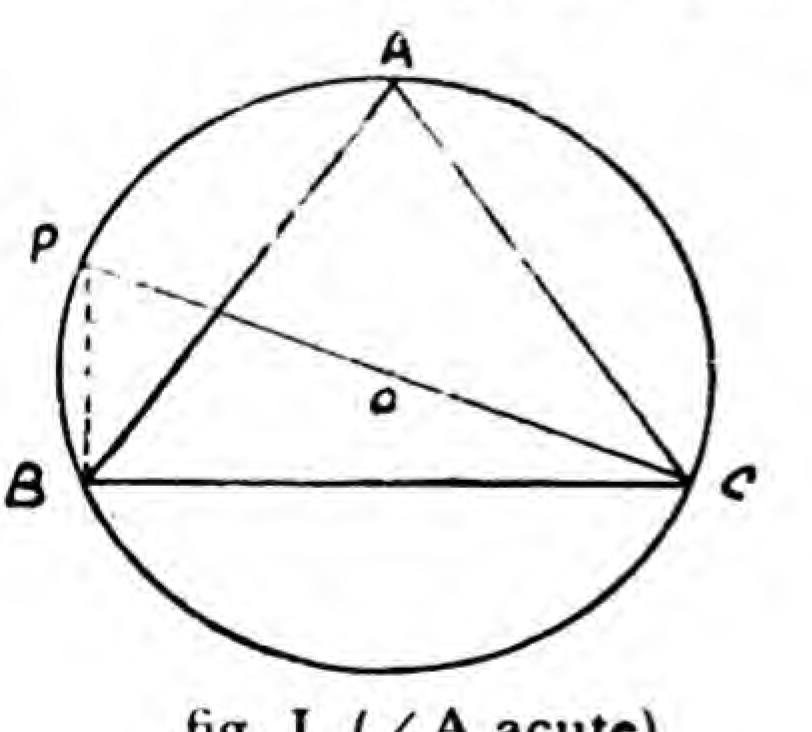


fig. I ($\angle$A acute)

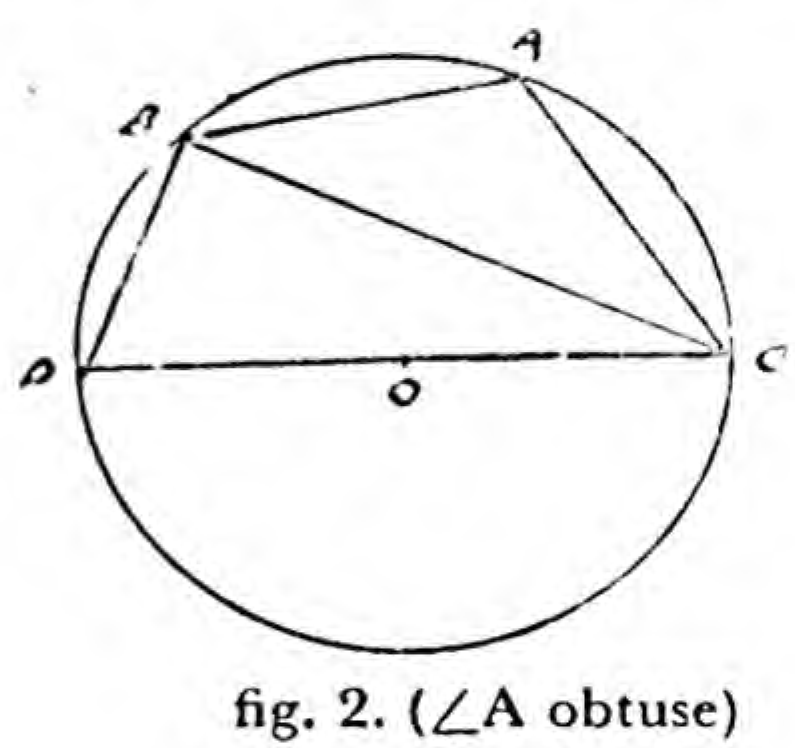


fig. 2. ($\angle$A obtuse)

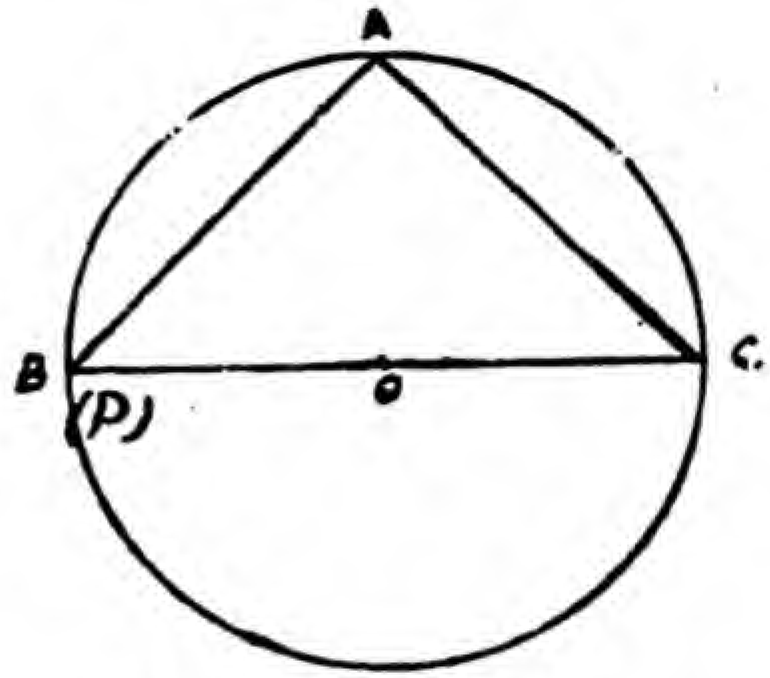


Fig. 3. ($\angle A=20^\circ$)

Let ABC be the triangle and O its circumcentre. Join CO and produce it to meet the circumcircle at P. Join BP.

Then $\angle CBP=90^\circ$ (angle in a semicircle)

In fig. 1, $\angle BPC=\angle BAC=A$ (angles in the same segment)

In fig. 2, $\angle BPC=180^\circ-\angle BAC=180^\circ-A$

(opposite angles of a cyclic quad. are supplementary)

$\therefore$ In both figs., $\sin \angle BPC=\sin A$

$\therefore$ from the rt-angled $\triangle$ PBC, $\dfrac{BC}{CP}=\sin \angle BPC=\sin A$

In fig. 3 also $\dfrac{BC}{CP}=1=\sin A$ $(\because A=90^\circ)$

Hence in all figs, $\dfrac{BC}{CP}=\sin A$, or $\dfrac{a}{2R}=\sin A$, $\therefore$ $R=\dfrac{\mathbf{a}}{\mathbf{2\sin A}}$

Similarly $R=\dfrac{b}{\mathbf{2\sin B}}=\dfrac{c}{\mathbf{2\sin C}}$.

Note. It follows that $\dfrac{a}{\sin A}=\dfrac{b}{\sin B}=\dfrac{c}{\sin C}=R$.

This is another proof of the Sine Formula, and the value of k in exs of Art. 2 chap. IX is 2R.

(*b*) **Another expression for R.**

To prove that in any triangle ABC,

$$R=\frac{abc}{4\triangle}. \qquad \text{(P. U. 1955)}$$

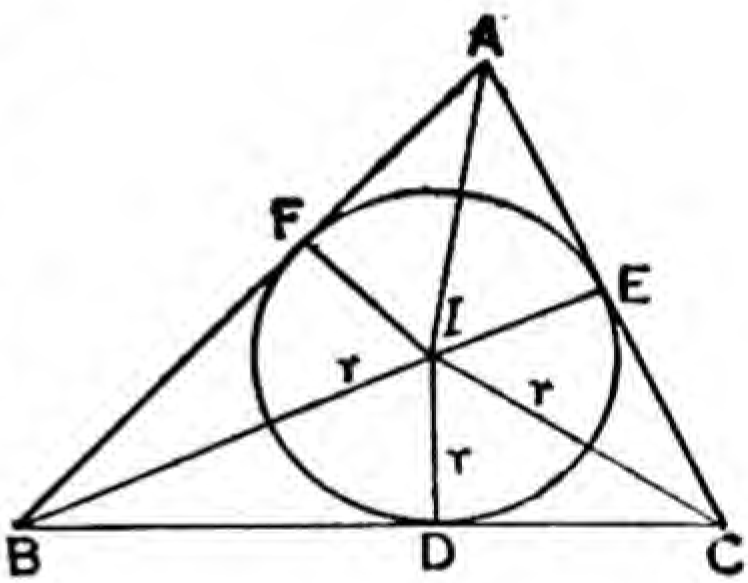
$\because R=\frac{a}{2\sin A}, \therefore \sin A=\frac{a}{2R}.$

$\therefore \triangle=\frac{1}{2}bc\sin A=\frac{1}{2}bc.\frac{a}{2R}=\frac{abc}{4R}$

$\therefore R=\frac{abc}{4\triangle}.$

3. (*a*) **To prove that in any triangle ABC,**

$$r=\frac{\triangle}{s}.$$



Let the bisectors of angles A, B and C meet in the point I, the incentre of the $\triangle$ ABC.

Draw ID, IE, IF perpendiculars to the sides; then

$ID=IE=IF=r$

$$\begin{aligned}\triangle ABC&=\triangle IBC+\triangle ICA+\triangle IAB\\&=\tfrac{1}{2}BC.r+\tfrac{1}{2}CA.r+\tfrac{1}{2}AB.r\\&=\tfrac{1}{2}r(a+b+c)=\tfrac{1}{2}r.2s=rs\end{aligned}$$

Hence $r=\frac{\triangle}{s}$.

(*b*) **To prove that in any triangle ABC,**

$$r=(s-a)\tan\frac{A}{2}.$$

Since the tangents from any external point to a circle are equal.

$\therefore$ AE=AF, BD=BF and CD=CE.

$$\begin{aligned}\therefore 2s&=(AE+AF)+(BD+BF)+(CD+CE)\\&=2AE+2BD+2DC\end{aligned}$$

$\therefore\ s = AE + BD + DC = AF + a$

$\therefore\ AE = s - a.$

From rt. angled $\triangle IAE$, $\dfrac{IE}{AE} = \tan\dfrac{A}{2}$

$$\therefore\ \frac{r}{s-a} = \tan\frac{A}{2}$$

$$\therefore\ r = (s-a)\tan\frac{A}{2}.$$

Similarly, $r = (s-b)\tan\dfrac{B}{2}$, and $r = (s-c)\tan\dfrac{C}{2}$.

4. To prove that

$$(i)\ r = \frac{a\sin\frac{B}{2}\sin\frac{C}{2}}{\cos\frac{A}{2}}$$

and (ii) $r = 4R\sin\dfrac{A}{2}\sin\dfrac{B}{2}\sin\dfrac{C}{2}$.

From the fig. in Art. 3,

$$a = BD + DC \qquad \text{.........(1)}$$

But $\dfrac{BD}{ID} = \cot\dfrac{B}{2}$ and $\dfrac{CD}{ID} = \cot\dfrac{C}{2}$

$\therefore\ BD = r\cot\dfrac{B}{2}$ and $CD = r\cot\dfrac{C}{2}$

$\therefore$ from (*i*), $a = r\cot\dfrac{B}{2} + r\cot\dfrac{C}{2}$

$$= r\left\{\frac{\cos\frac{B}{2}}{\sin\frac{B}{2}} + \frac{\cos\frac{C}{2}}{\sin\frac{C}{2}}\right\}$$

$$=r\frac{\left(\cos\frac{B}{2}\sin\frac{C}{2}+\sin\frac{B}{2}\cos\frac{C}{2}\right)}{\sin\frac{B}{2}\sin\frac{C}{2}}$$

$$=r\frac{\sin\left(\frac{B}{2}+\frac{C}{2}\right)}{\sin\frac{B}{2}\sin\frac{C}{2}}=r\frac{\cos\frac{A}{2}}{\sin\frac{B}{2}\sin\frac{C}{2}}$$

$$\left(\because \frac{B}{2}+\frac{C}{2}=90°-\frac{A}{2}\right)$$

$$\therefore \mathbf{r}=\frac{\mathbf{a}\sin\frac{\mathbf{B}}{2}\sin\frac{\mathbf{C}}{2}}{\cos\frac{\mathbf{A}}{2}}. \qquad \ldots\ldots(2)$$

Similarly $$\mathbf{r}=\frac{\mathbf{b}\sin\frac{\mathbf{C}}{2}\sin\frac{\mathbf{A}}{2}}{\cos\frac{\mathbf{B}}{2}}=\frac{\mathbf{c}\sin\frac{\mathbf{A}}{2}\sin\frac{\mathbf{B}}{2}}{\cos\frac{\mathbf{C}}{2}}$$

(*ii*) Again, since $a=2R\sin A$ (Art 2)

$$=2R.\ 2\sin\frac{A}{2}\cos\frac{A}{2}$$

$$=4R\sin\frac{A}{2}\cos\frac{A}{2}$$

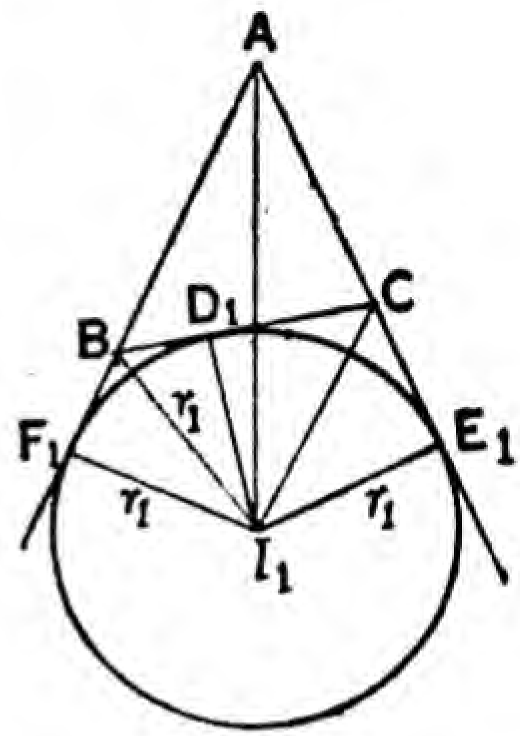
$\therefore$ substituting the value of a in (2) we get,

$$r=4R\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}.$$

Note. These results can also be proved by substituting in R. H. S. of the equations (*i*) and (*ii*) values of $\sin\frac{A}{2}$, $\cos\frac{A}{2}$ etc. in terms of sides as in **Ex. 4** hereafter.

5. (*a*) **To prove that in any triangle ABC,**

$$r_1 = \frac{\Delta}{s-a}.$$



Let I_1 be the ex-centre opposite to the angle A

Let D_1, E_1, F_1 be respectively the points of contact of the escribed circle with the side BC and the sides AC, AB, produced.

Join I_1D_1, I_1E_1, I_1F_1.

Then $I_1D_1 = I_1E_1 = I_1F_1 = r_1$

Join I_1A, I_1B, I_1C.

Now $\Delta\, ABC = \Delta I_1\, CA + \Delta\, I_1AB - \Delta I_1BC$

$= \frac{1}{2}\, CA.r_1 + \frac{1}{2}AB.\, r_1 - \frac{1}{2}\, BC.\, r_1$

$= \frac{1}{2}\, r_1\, (b+c-a) = \frac{1}{2}\, r_1\, (2s-2a)$

$= r_1(s-a)$

$$\therefore\ r_1 = \frac{\Delta}{s-a}.$$

Similarly $r_2 = \dfrac{\Delta}{s-b}$ and $r_3 = \dfrac{\Delta}{s-c}$

(*b*) **To prove that in any Δ ABC,**

$$r = s \tan \frac{A}{2}.$$

Since the tangents from an external point to a circle are equal.

$\therefore\ AE_1 = AF_1$, $BD_1 = BF_1$ and $CD_1 = CE_1$.

$\therefore\ 2s = AB + BC + CA$

$= AB + (BD_1 + D_1C) + CA$

$= AB + (BF_1 + CE_1) + AC$

$= (AB + BF_1) + (AC + CE_1)$

$= AF_1 + AE_1$

$= 2AE_1$ or $2\, AF_1$

$\therefore\ s = AE_1 = AF_1.$

From rt. angled $\triangle AE_1I_1$, $\frac{I_1E_1}{AE_1} = \tan\frac{A}{2}$

$$\therefore \frac{r_1}{s} = \tan\frac{A}{2}$$

$$\therefore \mathbf{r_1 = s \tan \frac{A}{2}}.$$

Similarly, $\mathbf{r_2 = s \tan \frac{B}{2}}$ **and** $\mathbf{r_3 = s \tan \frac{C}{2}}$.

To prove that $\mathbf{r_1 = (s-c) \cot \frac{B}{2} = (s-b) \cot \frac{C}{2}}$.

Here, $BD_1 = BF_1 = AF - AB = s-c$

$CD_1 = CE_1 = AE_1 - AC = s-b$

$\angle I_1BD_1 = \frac{1}{2}(180° - B) = 90° - \frac{B}{2}$

$\angle I_1CD_1 = \frac{1}{2}(180° - C) = 90° - \frac{C}{2}$

$\therefore$ from $\triangle I_1BD_1$, $\frac{I_1D_1}{BD_1} = \tan\left(90° - \frac{B}{2}\right)$

or $\frac{r_1}{s-c} = \cot\frac{B}{2}$

$$\therefore r_1 = (s-c)\cot\frac{B}{2}. \qquad \text{......(1)}$$

Again, from $\triangle I_1CD_1$, $\frac{I_1D_1}{CD_1} = \tan\left(90° - \frac{C}{2}\right)$

$$\therefore \frac{r_1}{s-b} = \cot\frac{C}{2}$$

$$\therefore r_1 = (s-b)\cot\frac{C}{2}. \qquad \text{......(2)}$$

Hence from (1) and (2), $r_1 = (s-c)\cot\frac{B}{2}$

$$= (s-b)\cot\frac{C}{2}.$$

Similarly we can prove that

$$r_2=(s-c)\cot\frac{A}{2}=(s-a)\cot\frac{C}{2}$$

$$r_3=(s-a)\cot\frac{B}{2}=(s-b)\cot\frac{A}{2}.$$

6. To prove (*i*) $r_1=\dfrac{a\cos\frac{B}{2}\cos\frac{C}{2}}{\cos\frac{A}{2}}$

(*ii*) $r_1=4R\sin\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}.$

From the fig. in Art 5,

(*i*) $a=BD_1+D_1C$ ......(1)

But $\dfrac{BD_1}{I_1D_1}=\cot\left(90^\circ-\dfrac{B}{2}\right)$

and $\dfrac{CD_1}{I_1D_1}=\cot\left(90^\circ-\dfrac{C}{2}\right)$

$\therefore\ BD_1=r_1\tan\dfrac{B}{2}$ and $CD_1=r_1\tan\dfrac{C}{2}$

$\therefore$ from (1), $a=r_1\left(\tan\dfrac{B}{2}+\tan\dfrac{C}{2}\right)$

$$=r_1\left\{\frac{\sin\frac{B}{2}}{\cos\frac{B}{2}}+\frac{\sin\frac{C}{2}}{\cos\frac{C}{2}}\right\}$$

$$=r_1\frac{\left(\sin\frac{B}{2}\cos\frac{C}{2}+\cos\frac{B}{2}\sin\frac{C}{2}\right)}{\cos\frac{B}{2}\cos\frac{C}{2}}$$

$$=r_1 \frac{\sin\left(\frac{B}{2}+\frac{C}{2}\right)}{\cos\frac{B}{2}\cos\frac{C}{2}}=\frac{r_1\cos\frac{A}{2}}{\cos\frac{B}{2}\cos\frac{C}{2}}$$

$$\left(\because \frac{B}{2}+\frac{C}{2}=90^\circ-\frac{A}{2}\right)$$

$$\therefore r_1=\frac{a\cos\frac{B}{2}\cos\frac{C}{2}}{\cos\frac{A}{2}} \qquad \ldots\ldots(2)$$

$$\text{Similarly } r_2=\frac{b\cos\frac{C}{2}\cos\frac{A}{2}}{\cos\frac{B}{2}}$$

$$\text{and } r_3=\frac{c\cos\frac{A}{2}\cos\frac{B}{2}}{\cos\frac{C}{2}}$$

(*ii*) Since $a=2R\sin A$ (Art. 2)

$$=2R\cdot 2\sin\frac{A}{2}\cos\frac{A}{2}$$

$$=4R\sin\frac{A}{2}\cos\frac{A}{2}$$

Putting this value of a in (2), we have

$$r_1=4R\sin\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}$$

$$\text{Similarly } r_2=4R\sin\frac{B}{2}\cos\frac{C}{2}\cos\frac{A}{2}$$

$$\text{and } r_3=4R\sin\frac{C}{2}\cos\frac{A}{2}\cos\frac{B}{2}.$$

Note :— The above can also be proved by substituting the values of $\sin\frac{A}{2}$, $\cos\frac{A}{2}$ etc. in terms of sides.

Ex. 1. The sides of a $\triangle$ are 13, 14, 15 ft. Calculate R, r, r_1, r_2, r_3. (M. U.)

$$s=\frac{13+14+15}{2}=\frac{42}{2}=21$$

$$\triangle=\sqrt{s(s-a)(s-b)(s-c)}=\sqrt{21\times8\times7\times6}=84$$

$$\therefore\ R=\frac{abc}{4\triangle}=\frac{13\times14\times15}{4\times84}=\frac{65}{8}.$$

$$r=\frac{\triangle}{s}=\frac{84}{21}=4,$$

$$r_1=\frac{\triangle}{s-a}=\frac{84}{8}=\frac{21}{2}$$

$$r_2=\frac{\triangle}{s-b}=\frac{84}{7}=12.$$

$$r_3=\frac{\triangle}{s-c}=\frac{84}{6}=14.$$

Ex. 2. Prove that

$$\frac{1}{r_1}+\frac{1}{r_2}+\frac{1}{r_3}=\frac{1}{r}$$ (J. & K. 1959)

$$\text{L. H. S.}=\frac{s-a}{\triangle}+\frac{s-b}{\triangle}+\frac{s-c}{\triangle}$$

$$=\frac{3s-(a+b+c)}{\triangle}=\frac{3s-2s}{\triangle}=\frac{s}{\triangle}=\frac{1}{r}.$$

Ex. 3. Prove that $r_1+r_2+r_3-r=4R$ (P. U. 1956)

$$\text{L. H. S.}=\frac{\triangle}{s-a}+\frac{\triangle}{s-b}+\frac{\triangle}{s-c}-\frac{\triangle}{s}$$

$$=\left(\frac{\triangle}{s-a}+\frac{\triangle}{s-b}\right)+\left(\frac{\triangle}{s-c}-\frac{\triangle}{s}\right)$$

$$=\triangle\cdot\frac{2s-a-b}{(s-a)(s-b)}+\triangle\cdot\frac{c}{s(s-c)}$$

$$=\triangle\cdot\frac{c}{(s-a)(s-b)}+\triangle\cdot\frac{c}{s(s-c)}$$

$$=c\triangle\left[\frac{1}{(s-a)(s-b)}+\frac{1}{s(s-c)}\right]$$

$$=c\triangle\left[\frac{s(s-c)+(s-a)(s-b)}{s(s-a)(s-b)(s-c)}\right]$$

$$=c\triangle\left[\frac{2s^2-s(a+b+c)+ab}{s(s-a)(s-b)(s-c)}\right]$$

$$=c\triangle\left[\frac{2s^2-2s^2+ab}{s(s-a)(s-b)(s-c)}\right]$$

$$=\frac{abc\triangle}{\triangle^2}=\frac{abc}{\triangle}=\frac{abc}{4\triangle}\times 4=4R$$

Ex. 4. Prove that $r_1=4R\sin\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}$.

(J. & K. U. 1949)

$$\text{R. H. S.}=4.\ \frac{abc}{4\triangle}\cdot\sqrt{\frac{(s-b)(s-c)}{bc}}$$

$$\times\sqrt{\frac{s(s-b)}{ca}}\cdot\sqrt{\frac{s(c-c)}{ab}}$$

$$=\frac{abc}{\triangle}\cdot\frac{s(s-b)(s-c)}{abc}=\frac{s(s-b)(s-c)}{\triangle}\times\frac{s-a}{s-a}$$

$$=\frac{s(s-a)(s-b)(s-c)}{\triangle(s-a)}=\frac{\triangle^2}{\triangle(s-a)}=\frac{\triangle}{s-a}=r_1$$

Or Thus: See Art. 6 part (*ii*)

EXERCISE 22.

With the usual notations in the $\triangle$ ABC, prove the following :—

1. $\frac{1}{ab}+\frac{1}{bc}+\frac{1}{ca}=\frac{1}{2Rr}$ (J. & K. U. 1961)

2. $Rr\,(\text{Sin } A+\sin B+\sin C)=\triangle$. (D. U. 1947)

3. $2R^2 \sin A \sin B \sin C=\triangle$.

4. $s=4R\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}$.

5. $\left(\frac{s}{a}-1\right)\left(\frac{s}{b}-1\right)\left(\frac{s}{e}-1\right)=\frac{r}{4R}$.

6. (*i*) $4\,R\,r\,s=abc$ (*ii*) $4\triangle=(b^2+c^2-a^2)\tan A$.

7. $\sin A+\sin B+\sin C=\frac{s}{R}$. (J. & K. U. 1958)

8. $a^2-b^2=2Rc\sin(A-B)$.

9. $\frac{1}{r_1^2}+\frac{1}{r_2^2}+\frac{1}{r_3^2}+\frac{1}{r^2}=\frac{a^2+b^2+c^2}{\triangle^2}$ (P. U. 1952)

10. $r_2r_3+r_3r_1+r_1r_2=s^2$. (D. U. 1950)

11. $(r_1+r_2)\tan\frac{C}{2}=(r_3-r)\cot\frac{C}{2}=c$. (P. U. 1944)

12. (*i*) $\triangle^2=rr_1r_2r_3$. (J. & K. U. 1956)

(*ii*) $a=(r_2+r_3)\sqrt{rr_1/r_2r_3}$ (P. U. 1954) (D. U. 1953)

13. (*i*) $rr_1=r_2r_3\tan^2\frac{A}{2}$. (Allahabad U).

(*ii*) $(r_1-r)(r_2-r)(r_3-r)=4Rr^2$. (M. U. 1944)

(*iii*) $r_1+r_2+r_3=4R+r$ (D. U. 1956)

14. (*i*) $\Delta=r^2 \cot\frac{A}{2}\cot\frac{B}{2}\cot\frac{C}{2}$ (J. & K. U. 1952 S)

(*ii*) $\Delta=\frac{2abc}{a+b+c}\cdot\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}$. (Cal. U:)

(*iii*) $1-\sin^2\frac{A}{2}-\sin^2\frac{B}{2}-\sin^2\frac{C}{2}=\frac{r}{2R}$

(P. U. 1955)

15. (*i*) $\frac{r_1+r_2}{1+\cos C}=\frac{r_2+r_3}{1+\cos A}=\frac{r_3+r_1}{1+\cos B}$.

(*ii*) $\frac{r_1-r}{a}+\frac{r_2-r}{b}=\frac{c}{r_3}$. (P. U. 1936 S)

16. $\cos A+\cos B+\cos C=1+\frac{r}{R}$ (J. & K. U. 1953)

[**Hint.** Use the Identity $\cos A+\cos B+\cos C$

$=1+4\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$ and Art. 4 (*ii*)]

17. $a\cot A+b\cot B+c\cot C=2R+2r$. (K. U. 1949)

[**Hint.** Put $a=2R\sin A$ etc. then it reduces to Q. 16]

18. Determine the radius of the circumscribed circle of the triangle whose sides are 5″, 7″, 9″. (P. U.)

19. Find the area of the inscribed circle of the triangle whose sides are 4″, 5″, 7″ respectively. (M. U.)

20· The sides of a triangle are 16, 20, 33 ft. Find the radius of the escribed circle corresponding to the greatest angle. (P. U. 1939)

21. If $a=53{\cdot}6$, $b=64{\cdot}3$, $c=52{\cdot}5$, calculate the area of the inscribed circle of the triangle. (P. U. 1945)

22. If $a=234{\cdot}5$, $b=317$ and $c=341{\cdot}3$; find the radius of the escribed circle touching the side BC. (P. U. 1945)

23. Prove that $R=\dfrac{abc\ (\cot A+\cot B+\cot C)}{a^2+b^2+c^2}$ (D.U. 1935)

24. Show that $a \cos A+b \cos B+c \cos C=4R \sin A \sin B \sin C$. (P. U. 1955)

25. Show that $\dfrac{1}{s-a}+\dfrac{1}{s-b}+\dfrac{1}{s-c}-\dfrac{1}{s}-\dfrac{4R}{\triangle}$.

26. Find the radius of the escribed circle of a triangle ABC corresponding to the angle B. (P. U. 1956)

CHAPTER 11

Logrithms

1. The word Logarithm consists of two parts (*i*) *log* (meaning : rule or plan) and (*ii*) *arithm* (meaning arithmatic). Consequently. Logarithm means "a rule to shorten Arithmatic". Its invention by **John Napier** in 1614 was a great landmark in the history of the development of Mathematics. in the words of **Laplace,** "*Logarithm reduces to a few days the labour of many months and so doubles, as it were, the life (of a Mathematician) besides freeing him from the errors and disgust inseparable from long calculation*". A brief account of the properties of logarithms and their uses is given below :—

2. We known that $2^3=8$, where 2 is the base and 3 is the index of the power. This is also expressed by saying that logarithm of 8 to the base 2 is 3 and is written as $log_2 8=3$. Similarly $4^3=64$ is written as $\log_4 64=3$; and in general if $a^x=N$, then $\log_a N=x$.

Definition :—The logarithm of a number to a given base is the index of the power to which the base must be raised in order to make it equal to the given number

Ex 1. Find the logarithm of 81 to the base 3.

Let $\log_3 81=x$,

then $3^x=81$

or $3^x=3^4 \quad \therefore x=4$

Ex. 2. Find the logarithm of 64 to the base 16.

Let $\log_{16} 64=x$

$\therefore 16^x=64$

or $(2^4)^x=2^6$

or $2^{4x}=2^6$, $\therefore 4x=6$ *i. e.* $x=\frac{3}{2}$.

3. **Particular cases.**

(1) $\because a^0=1, \therefore \log_a 1=0$.

i. e. **log 1 to any base=0.**

(2) $\because a^1 = a, \therefore \log_a a = 1$

i. e. **log of any number to the same number as base is unity.**

(3) $\because a^{-\infty} = \dfrac{1}{a^{\infty}} = 0$, where $a > 1$

$\therefore \log_a 0 = -\infty$; where $a > 1$

i. e. $-\infty$ is the logarithm of 0 to base a $(a > 1)$

(4) $\because a^x = -7$ is not satisfied by any real value of $x (a > 0)$

$\therefore$ The logarithm of a negative number to any positive base is imaginary.

i. e. **a negative number has no logarithm.**

4. Corresponding to the 3 laws of indices :

(*i*) $a^m \times a^n = a^{m+n}$

(*ii*) $\dfrac{a^m}{a^n} = a^{m-n}$

$(a^m)^n = a^{mn}$

we have the 3 fundamental laws of logarithms, viz.

(*i*) $\mathbf{\log_a mn = \log_a m + \log_a n}$

(*iii*) $\mathbf{\log_a \dfrac{m}{n} = \log_a m - \log_a n}$

(*iii*) $\mathbf{\log_a m^n = n \log_a m.}$

we shall prove these in the following articles.

5. **To prove that $\log_a mn = \log_a m + \log n$.**

i. e. the log of the product of two factors is equal to the sum of the logs of the factors

Proof. Let $\log_a m = x$, and $\log_a n = y$

so that $m = a^x$, and $n = a^y$

$\therefore mn = a^x . a^y = a^{x+y}$

$\therefore \mathbf{\log_a mn = x + y = \log_a m + \log_a n}$.

Note 1. If should be carefully noted that $\log_a (m+n)$ is not equal to $\log_a m+\log_a n$.

Note 2. By a similar method of proof the formula can be extended to any number of factors *i. e.* $\log_a (mnp\ldots\ldots\ldots\ldots)$ $=\log_a m+\log_a n+\log_a p+\ldots\ldots\ldots\ldots$

6. To prove that $\log_a \frac{m}{n}=\log_a m-\log_a n$.

i. e. the logarithm of a fraction is equal to the logarithm of the numerator minus the logarithm of the denominator.

Proof. Let $\log_a m=x$ and $\log_a n=y$

so that $m=a^x$ and $n=a^y$.

$$\therefore \quad \frac{m}{n}=\frac{a^x}{a^y}=a^{x-y}$$

$$\therefore \log_a \frac{m}{n}=x-y=\log_a m-\log_a n.$$

7. To prove that $\log_a m^n=n \log_a m$

i. e. the logarithm of any power of a number is equal to the index of the power into the logarithm of the number.

Proof. Let $\log_a m=x$, then $m=a^x$

$\therefore m^n=(a^x)^n=a^{nx}$

$\therefore \log_a m^n=nx=n\log_a m.$

Note. It follows that in working with logarithms of numbers, the process of

(*a*) Multiplication is replaced by addition,

(*b*) Division „ „ „ subtraction,

(c) Involution „ „ „ multiplication.

(*d*) Involution „ „ „ Division.

Ex. Show that $\log_a \frac{x^4y^5}{z^7t^9}=4 \log_a x+5 \log_a y-\log_a z$
$-9 \log_a t$

$$\log \frac{x^4y^5}{z^7t^9}=\log_a (x^4y^5)-\log_a (z^7t^9) \qquad \text{(Art. 6)}.$$

$=\log_a x^4+\log_a y^5-(\log_a z^7+\log_a t^9)$ **(Art. 5)**

$=4\log_a x+5\log_a y-7\log_a z-9\log_a t.$ **(Art. 7)**

8· Transformation of bases of logarithms.

To prove that $\log_a m=\log_b m\times\log_a b$

Proof. (*i*) Let $\log_b m=x$ and $\log_a b=y$

then $m=b^x$ and $b=a^y$

$\therefore\ m=(a^y)^x=a^{xy}$

$\therefore\ \mathbf{\log_a m=xy=\log_b m\times\log_a b}$

Cor. (*i*) $\mathbf{\log_b m=\dfrac{\log_a m}{\log_a b}}$.

Note. This is the formula **for change of base. It enables** us to find the logarithm of m to the base b, **when** the logarithm of m and b each to the base a is known.

Cor. (*ii*). $\log_b a\times\log_a b=1$

Proof. In the above formula, put $m=a$

$\therefore\ \log_a a=\log_b a\times\log_a b.$

$\because 1=\log_b a\times\log_a b$

Cor. (*iii*) $\mathbf{\log_b a=\dfrac{1}{\log_a b}}$.

Ex Given $\log_{10}3=\cdot4771$, find $\log_3 10$

Sol. $\log_3 10=\dfrac{\log_{10}10}{\log_{10}3}=\dfrac{1}{\cdot4771}=2\cdot096.$

9. Two systems of logarithms are in use :—

1. **Common Logarithms.** When the number **10 is taken** as the base of logarithms the system is known **as Common Logarithms,** so called because it is used commonly **in all** practical calculations.

2. **Natural Logarithms.** In all theoretical calculations in higher Mathematics logarithms to the base e **are used,** e being the sum of the infinite series :—

$$1+1+\frac{1}{\lfloor 2}+\frac{1}{\lfloor 3}+\ldots\ldots\ldots \text{to } \infty, =2{\cdot}71828 \text{ approx.})$$

These logarithms are known as Natural or Napierian Logarithms after the name of **Napier** who first calculated these to base e.

Note. In this book only the **common logarithms** will be used. In writing such logarithms the base will be omitted, so that $\log_{10}28$ is written simply as log 28.

10. Two parts of Common Logarithm. The logarithm of a number is not always integral. Thus since $10^2=100$ and $10^3=1000$, the logarithm of a number like 534 lying between 100 and 1000 will be between 2 and 3 and will be $=2+$ *a positive proper fraction.*

Similarly since $\log {\cdot}01=\log 10^{-2}=-2$

and $\log {\cdot}001=\log^{-3}=-3$

$\therefore$ logarithm of any number like ·003, lying between ·01 and ·001 is a negative number greater than -3 but less than -2 and may be written $= -3 +$ *a positive proper fraction.* Thus the common logarithm of a number consists of two parts (*i*) **integral and** (*ii*) **fractional.** The integral part is called the **characteristic** and the fractional part, which *must be positive*, is called the **mantissa**..

Ex. The logarithms of two numbers are

(*i*) 2·4771 and (*ii*) $-4{\cdot}235$

Find the characteristics and the mantissae.

Sol. (*i*) Here 2 being the integral part is the characteristic and .4771 being the positive fractional part is the mantissa.

$$(ii)\ -4{\cdot}236=-4-{\cdot}236=-4-1+1-{\cdot}236$$
$$=-5+(1-{\cdot}236)=-5+{\cdot}764$$

Here -5 is the characteristic and ·764 which is the positive fractional part is the mantissa.

Note. 1. To make the decimal part positive we must subtract 1 from the integral part and add 1 to the decimal part.

Note. 2. For brevity $-5+\cdot764$ is written as $\bar{5}.764$. The horizontal line, called the **bar**, over 5 denotes that the integral part alone is negative while $\cdot764$ is positive but in $-5\cdot764$ both 5 and $\cdot764$ are negative. $\bar{5}$ is read as 5 bar.

11. To show that the characteristic of the logarithm of any number N can be written down by inspection.

Case I. Let the number N be >1, having n digits in its integral part, then since.

(*i*) a number has 1 digit in its integral part when it lies between 1 and 10 *i e.* $(10)^\circ$ and $(10)^1$

(*ii*) a number has 2 digits in its integral part. when it lies between 10 and 100 *i.e.* between 10^1 and 10^2.

Therefore, the number **N** which has n digits in its integral part, lies between 10^{n-1} and 10^n.

$$(n-1) + a \text{ fraction.}$$

$$\therefore \quad N=10$$

$$\therefore \quad \log N=(n-1)+a \text{ fraction.}$$

Hence the characteristic is $\mathbf{n-1}$.

Therefore the characteristic of the logarithm of a number greater than unity is one less than the number of digits in the integral part of the number.

Case II. Let N be positive and <1, having n zeros immediately after the decimal point, then since.

(*i*) a number has one zero immediately after the decimal point (as in $\cdot07$) when it lies between .01 and .1 *i. e.* between 10^{-2} and 10^{-1}

(*ii*) a number has 2 zeros immediately after the decimal point (as in $\cdot003$) when it lies between $\cdot001$ and $\cdot01$ *i. e.* between 10^{-3} and 10^{-2} and so on.

Therefore the number N which has n zeros immediately after the decimal point lies between $10^{-(n+1)}$ and 10^{-n}.

$-(n+1)+a$ fraction

$\therefore N=10$

$\therefore \log N=-(n+1)+a$ fraction

Hence the characteristic is $-(n+1)$.

Therefore the characteristic of the logarithm of a number less than unity is negative and numerically greater by one than the number of cyphers immediately after the decimal point.

Thus the characteristics of the logarithms of the numbers 6745, 67·45 and 648·3 are 3, 1 and 2 respectively, and the characteristics of the logarithms of the numbers ·0043, ·02503 and ·73435 are—3,—2, and — 1 respectively.

12. To show that mantissae of the logarithms of all numbers consisting of the same digits arranged in the same order but differing in the position of the decimal point are the same.

If any two numbers have the same digits arranged in the same order, they differ only in the position of the decimal point, so that one number must be equal to the other multiplied by some power of 10. Hence logarithms differ by an integer *i. e.* in their characteristic only.

For example, if $\log 6{\cdot}804 = {\cdot}8328$,

$$\text{then } \log 68{\cdot}04 = \log (6{\cdot}804\times 10) = \log 6{\cdot}804 + \log 10$$
$$= {\cdot}8328 + 1 = 1{\cdot}8328,$$

$$\text{and } \log {\cdot}06804 = \log (6{\cdot}804 \times 10^{-2}) = \log 6{\cdot}804 + \log 10^{-2}$$
$$= \bar{2}.8328$$

13. Advantages of common System of Logarithms :—

The common system of logarithms has the following two important advantages.

1. The characteristic of the logarithm of any number can be found out by inspection
2. The mantissae of the logarithms of all numbers consisting of the same digits arranged in the same order (*i. e.* of numbers differing only in the position of the decimal point) are the same.

Ex. 1. If $\log 7645 = 3{\cdot}8834$ find out the logarithms of $7{\cdot}645$, $\cdot07645$ and $\cdot0007645$.

Since these numbers consist of the same digits in the same order, their logarithms will have the same mantissae but different characteristics.

Hence $\log 7.615 = \cdot8834$

$\log \cdot07645 = \bar{2}.8834$

$\log \cdot0007645 = \bar{4}.8834$

Ex. 2. Give $\log 2 = \cdot3010$, find

(*i*) $\log \cdot0016$ **(M. U.)**

(*ii*) $\log \cdot0005$ **(M. U.)**

Sol. (*i*) Since $\cdot0016 = \dfrac{16}{10000} = \dfrac{(2)^4}{(10)^4}$

$\therefore \log \cdot0016 = 4\log 2 - 4\log 10 = 4\log 2 - 4$

$= -4 + 4(\cdot3010) = -4 + 1{\cdot}2040 = \bar{3}.2040$

(*ii*) $\cdot0005 = \dfrac{5}{10000} = \dfrac{5\times 2}{10000\times 2} = \dfrac{1}{10^3\times 2}$

$\therefore \log \cdot0005 = \log 1 - 3\log 10 - \log 2 = -3 - \cdot3010$

$= -3 - 1 + 1 - \cdot3010 = \bar{4}.6990.$

Ex. 3. Given $\log 2 = \cdot30103$, find

(*i*) the number of digits in 2^{56} and (*ii*) the position of the first significant figure in 2^{-25}.

Sol. (*i*) $\log 2^{56} = 56\log 2 = 56 \times \cdot30103 = 16{\cdot}85768$

Thus the characteristic is 16

Hence the number of digits $= 16 + 1 = 17$.

(*ii*) $\log 2^{-25} = -25\log 2 = -25 \times \cdot30103 = -7{\cdot}52575$

$= \bar{8}.47425$

Hence the characteristic is -8.

$\therefore$ number of zeros immediately after the decimal point $= 8 - 1 = 7$.

Thus the first significant figure is in the 8th place after the decimal point.

Ex. 4. Given that $\log 3 = \cdot 4771$, $\log 7 = \cdot 8451$ and $\log 11 = 1\cdot 0414$, solve the equation
$3^x \times 7^{2+x} = 11^{x+5}$. (P. U. 1943 S)

Sol. Taking logarithms we have

$$\log 3^x + \log 7^{2x+1} = \log 11^{x+5}$$

$$\therefore\ x \log 3 + (2x+1) \log 7 = (x+5) \log 11$$

$$\therefore\ x (\log 3 + 2 \log 7 - \log 11) = 5 \log 11 - \log 7$$

$$\text{or}\quad x = \frac{5 \log 11 - \log 7}{\log 3 + 2 \log 7 - \log 11}$$

$$= \frac{5\cdot 2070 - \cdot 8451}{\cdot 4771 + 1\cdot 6902 - 1\cdot 0414}$$

$$= \frac{4\cdot 3619}{1\cdot 1258} = 3\cdot 8 \text{ nearly.}$$

Note. When the logarithms of certain numbers are given in the question, only those and no others should be used (excepting those of 1 and powers of 10 which are obvious).

Exercise 23.

1. (*a*) Write down the characteristics of the common logarithms of the following numbers :
   (*i*) 3576 (*ii*) 478965 (*iii*) ·5487 (*iv*) ·00345
   (*v*) ·0000032

   (*b*) Find the value of :—
   (*i*) $\log_2 256$ (*ii*) $\log_8 128$ (*iii*) $\log_{81} 729$

2. If $\log 8543 = 3\cdot 9317$ find the logarithms of
   8·543, 854·3, ·008543.

3. If a, b, c, d are any positive numbers prove that

$$\log \frac{a^2}{bc} + \log \frac{b^2}{ac} + \log \frac{c^2}{ab} = 0$$

4. Given $\log 2 = \cdot 3010$, find the values of
   (*i*) $\log 6\cdot 4$ (*ii*) $\log \cdot 0025$ (*iii*) $\log (6\cdot 4)^{-3}$

5. Give log $3 = \cdot 4771$, find the number of digits in 3^{62} and the position of the first significant figure in 3^{-32} (P. U. 1951)

6. If log $2 = \cdot 30103$, find the number of digits in 2^{23}. (J. & K. U. 1958)

7. Given log $2 = \cdot 3010$, log$3 = \cdot 4771$, calculate to two decimal places the values of

(*i*) $\log_8 27$ (*ii*) $\log_2 10$. (M. U.)

8. If $a^{3-x} \times b^{5x} = a^{x+5} \times b^{3x}$ thow that

$$x \log \frac{b}{a} = \log a$$

(P. U.)

9. If a, b, c are in G. P., show that $\log_a n$, $\log_b n$, $\log_c n$ are in H·P. (P. U. 1931)

10. Given log $2 = \cdot 3010$, log $3 = \cdot 4771$ and log $7 = \cdot 8451$, solve the equations :—

$$(i)\ 2^x \times 3^{2x+1} = 7^{4x+3} \quad (ii)\ 2^x . 3^{x+4} = 7^x$$

Use of Logarithm Tables

14. (*a*) **To find the logarithm of a number consisting of 4 significant figures.**

The *characteristic* of the logarithm of a number is written down by inspection by Art. 11.

The *mantissa* alone is obtained from the tables.

The logarithm tables, on the first two pages, given at the end of the book consist of 3 parts :—

1. The extreme left column contains numbers from 10 to 99. These two digits correspond to the **first two** significant figures of the number.

2. Next 10 columns are headed by 0, 1, 2, 3......9. These correspond to the *third figure* of the number.

3. Next 9 columns called the '**mean difference**' columns headed by figures 1, 2, 3,............9. These correspond to the *fourth figure* in the number.

To find the mantissa of the logarithm of a number consider the number as if it has no decimal point in it. Then pick out the horizontal row containing the *first two* significant figures in the first column and the vertical column corresponding to the *third figure* and note down the number at their junction. Add to this number the number in the same row under the mean difference column headed by the *fourth figure*.

Note 1. The mantissae are given correct to four decimal places with the decimal point omitted (for convenience of printing.)

For example, let us find log 5243.

The characteristic in this case is 3.

For mantissa we note that the first two figures from the left form the number 52, the third figure is 4 and the 4th is 3.

Now looking in the horizontal row containing 52 and under the column headed by 4 we find the number 7193 at the junction. Passing along this row and under the mean difference column headed by 3, we find the number 2.

$\therefore$ the mantissa$=7193+2=7195$ (omitting decimal pt.)

$\therefore$ mantissa$=\cdot 7195$.

Hence logarithm$=3\cdot 7195$.

Note. To find the mantissa of the logarithm of a number which does not contain four significant figures we add zeros to the right of the number until it contains four figures.

Ex. Find log 6.

Here characteristic is zero.

Adding zeros to the right of 6, we get 6000. From the tables, the mantissa$=\cdot 7782$.

Hence log $6=0.7782$

(*b*) To find the number whose logarithm is given.

This is done with the help of the table of **anti-logarithms,**

The first column in the tables contains numbers from ·00 to ·99 and the other columns are similar to those of logarithm tables.

We do not consider the characteristic at first and look for the *first two* significant figures of the mantissa in the first column of the table. Moving along the row containing these and under the column headed by the *third figure* of the mantissa we find the number at the junction. To this we add the number at the junction of the same row and under the mean difference column headed by the *fourth figure* of the mantissa.

The position of the decimal point is determined from the characteristic as it gives us the number of digits in the integral part, or the number of zeros immediately after the decimal point, of the required number.

Ex. Find x, when $\log x = 1{\cdot}8762$.

We do not consider the characteristic at first.

Looking for the number ·87 in the first column and moving along the row containing it under the column headed by 6 we find the number 7316. In the same row under the mean difference column headed by 2 we obtain the number 3. Adding 3 to 7516 we get 7519.

Since the characteristic is 1, the decimal point will be placed after 2 figures. Counting the figures of 7519 from the left we have $x = 75{\cdot}19$.

Ex. 1. Find the value of $(2{\cdot}709)^{\frac{1}{5}} \times (1{\cdot}2387)^{\frac{1}{7}}$

Let $x = (2{\cdot}709)^{\frac{1}{5}} \times (1{\cdot}2387)^{\frac{1}{7}}$

Taking logarithms,

$$\log x = \tfrac{1}{5}\log(2{\cdot}709) + \tfrac{1}{7}\log(1{\cdot}2387)$$
$$= \tfrac{1}{5}[{\cdot}4328] + \tfrac{1}{7}[{\cdot}0930]$$
$$= {\cdot}0866 + {\cdot}0133 = 0{\cdot}999$$

From anti-log tables, $x = 1{\cdot}158$.

Ex. 2. Rs. 100 invested in a post-office cash certificate becomes Rs. 150 after 12 years. Find the rate percent of compound interest.

The formula for Amount in compound interest is

$$P\left(1+\frac{r}{100}\right)^t=A$$

Here A=150, P=100, t=12 yrs.

$$\therefore\ 100\left[1+\frac{r}{100}\right]^{12}=150$$

or $$\left[1+\frac{r}{100}\right]^{12}=\frac{3}{2}$$

Taking logarithms, $12\log\left[1+\frac{r}{100}\right]=\log 3-\log 2$

$$\therefore\ \log\left[1+\frac{r}{100}\right]=\frac{\log 3-\log 2}{12}=\frac{\cdot4771-\cdot3010}{12}$$

$$=\frac{\cdot1761}{12}=\cdot0147$$

$$\therefore\ 1+\frac{r}{100}=1\cdot035$$

$\therefore$ $r=3\cdot5\%$

Exercise 24.

1. Find the logarithms of : (*i*) 45·93 (*ii*) ·0927 (*iii*) $(2\cdot7)^{\frac{1}{2}}$

2. Find the antilogarithms of : (*i*) 0·4962, (*ii*) 2·0930 (*iii*) $\overline{2}\cdot4328$

3. Find the value of : (*i*) $(58\cdot95)^{\frac{1}{3}}$

(*ii*) $\dfrac{(435)^3\sqrt{\cdot056}}{(380)^4}$ (P U.) (*iii*) $\dfrac{(3\cdot142)^3(\cdot078)^{\frac{1}{3}}}{(\cdot005)^{\frac{1}{4}}}$ (P. U.) (1949)

(*iii*) $\dfrac{(21\cdot65)^{\frac{3}{4}}\times(0\cdot75)^{\frac{9}{7}}}{(12\cdot56)^{\frac{7}{5}}\times(75\cdot18)^{\frac{4}{5}}}$ (P. U. 1954)

4. Find the 5th root of 256·4.

5. The post office 5 year cash certificates for Rs. 500 are obtainable at an issue price of Rs. 440, 10 as. Find the rate per cent of compound interest. (P. U. 1940)

15. (*a*) **Tables of natural Trigonometric-ratios.**

Along with the tables of logarithms and anti-logarithms are given the tables of Natural T-ratios. Since the T-ratios of any angle can be expressed by T-ratios of an angle lying between $0°$ and $90°$, the tables give T-ratios of angles between $0°$ and $90°$ only. Each table is divided into 3 sets of columns :

(1) The extreme left column headed by Degrees.

(2) The next 10 columns headed by $0'$, $6'$, $12'$...,$54'$.

(3) The next 5 columns headed by 1, 2, 3, 4, 5 and known as **mean difference columns**. The numbers in these columns are to be added in the case of sine and tangent and subtracted in the case of cosine, because $\sin\theta$ and $\tan\theta$ increase with θ whereas $\cos\theta$ decreases as θ increases.

The cosecant, secant, or cotangent of an angle can be calculated by taking the reciprocals of sine, cosine or tangent of that angle respectively.

Ex. 1. Find (*i*) $\sin 39° 28'$ (*ii*) $\sin 136° 20'$

Turning to the tables of Natural Sines we spot 39 in the extreme left column of degrees. In the horizontal line containing 39 and under $24'$ (nearest multiple of $6'$ in $28'$) we find the number 6347 at the junction. In the same row and under the *mean difference* column headed by $4'$ we find the number 9 at the junction. Adding 9 to 6347 we get 6356. Prefixing the decimal point shown in that row under the column headed by $0'$, we get $\sin 39° 28' = \cdot 6356$.

$$(ii)\ \sin 136° 20' = \sin (180° - 43° 40') = \sin 43° 40'$$
$$= \cdot 6904$$

Ex. 2. Find $\cos 35° 47'$.

Turning to the table of Natural Cosines, in the horizontal line of $35°$ and under $42'$, we find the number 8121 at the junction. In the same horizontal line and under 5 in the mean difference column, we get the number 8. Subtracting 8 from 8121 we get 8113. Prefixing the decimal point we have $\cos 35° 47' = \cdot 8113$.

Note. We can also get cos 35° 47′ by finding the sine of the comlimentary angle 54° 13′.

Thus from Natural Sine tables sin 54° 13′ = ·8114.

Ex. 3. Find tan 64° 45′.

Turning to the table of Natural Tangent, in the horizontal line of 64° and under 42′, we find the number 1155 at the junction. In this same horizontal line and under 3′ in the mean difference columns, we get the number 47. Adding 47 to 1155 we get 1201. Prefixing the *integer* 2 before. ·1202 as given in that row under the column headed by 0 we get tan 64°45′ = 2·1202.

Note. cot θ can be found with the help of the tangent of the compliment of θ *i. e.*, from tan $(90° - \theta)$.

(*b*) **To find the angle when any one of its T-ratios is given.**

Ex. 4. If sin θ = ·51, find θ.

Turning to the table of Natural Sines, the number nearest to ·51 (*i. e.* ·5100) is ·5090 which lies at the junction of 30° and 36′. The remaining difference of 10 is to be looked for in the mean difference columns. In these columns 10 occurs under 4′. Adding 4′ we get $\theta = 30° \ 40'$.

Ex. 5. If cos θ = ·5085, find θ.

Turning to the table of Natural Cosines, the number nearest to ·5085 is ·5075 which lies at the junction of 50° and 30′. The remaining difference of 10 is to be looked for in the mean difference columns. In these colums 10 occurs under 4′. Subtracting 4′ from 59° 30′ we get $\theta = 59° \ 26'$.

Ex. 6. If tan θ = 1·6920, find θ.

Turning to the table of Natural Tangents the number nearest to 1·6920 is 1·6909 which lies at the junction of 59° and 24′. The remaining difference of 11 is to be seen in the mean difference columns. In these columns 11 occurs under 1′. ∴ Adding 1′ to 24′ we get $\theta = 59° \ 25'$.

16. Tables of Logarithms of Trigonometric ratios.

These tables give the logarithms of the trigonometric ratios of all angles from $0°$ to $90°$ and are consulted in the same way as the tables of Natural T-ratios.

Ex. 1. Find (*i*) log sin $49° 26'$ and (*ii*) log sin $156° 44'$.

(*i*) In the table of logarithms of sines, along the horizontal line containing $49°$ and under the column $24'$ we find the number 8804. In the same line and under the mean difference column headed by $2'$ the number is 2. Adding 2 to 8804 we get 8806, which is the mantissa of log sin $49° 26'$. The characteristic $\bar{1}$ is shown only in the column under $0'$ in the horizontal row containing $49°$. Hence log sin $49° 26'$ $=\bar{1}\cdot8804$.

(*ii*) log sin $156° 44' =$ log sin $(180° - 23° 16')$
$=$ log sin $23° 16' = \bar{1}\cdot5966$.

Ex. 2. Find log cos $36° 34'$.

In the tables of logarithms of cosines, along horizontal line containing $36°$ and under the column $30'$ we find the number 9052. In the same line under the mean difference column headed by $4'$, the number is 4. Subtracting 4 from 9052 we get 9948 which is the mantissa of log cos $36° 34'$. The characteristic $\bar{1}$ is shown only in the column under $0'$ in the horizontal row containing $36°$.

$\therefore$ log cos $36° 34' = \bar{1}\cdot9048$.

Ex. 3. Find log tan $40° 45'$.

In the tables of logarithms of tangents, along the horizontal line containing $40°$ and under the column $42'$ we find the number 9346. In the same line under the mean differenoe column headed by $3'$ the number is 8. Adding 8 to 9346 we get 9354 which is the mantissa of log tan $40° 45'$. The eharacteristic $\bar{1}$ is shown only under $0'$ in the row eontaining $40°$.

$\therefore$ log tan $40° 45' = \bar{1}\cdot9354$.

17. Tabular logarithms. Since the sine and cosine of an angle are always less than one, the characteristics of their logarithms are always negative. The same is the case with

the tangent of an angle less than 45°. To avoid the inconvenience of printing bars over the characteristic, in some tables the logarithms of the T-ratios are increased by 10 and are called **Tabular Logarithms.**

The Tabular Logarithm is denoted by the capital letter L instead of log. Thus :—

(*i*) $L \sin 49° 26' = 10+\log \sin 49° 26' = 10+\bar{1}{\cdot}8806$
$= 9{\cdot}8806.$

(*ii*) $L \cos 36° 50' = 10+\log \cos 36° 50'$
$= 10+\bar{1}.9033$
$= 9{\cdot}9033$

(*iii*) $L \tan 44° 22' = 10+\log \tan 44° 22'$
$= 10+\bar{1}{\cdot}9904 = 9{\cdot}9904.$

Again, let us find θ, given :—

(*i*) $L \sin \theta = 9{\cdot}62,$

(*ii*) $L \cos \theta = 9{\cdot}2121$

(*iii*) $L \tan \theta = 9{\cdot}9422.$

(*i*) $L \sin \theta = 10+\log \sin \theta = 9{\cdot}62$
$\therefore \log \sin \theta = 9\ 62-10$
$= -1+{\cdot}6200$
$= \bar{1}.6200$

$\therefore \theta = 24° 38'.$

(*i*) $L \cos \theta = 10+\log \cos \theta = 9{\cdot}9121$
$\therefore \log \cos \theta = 9{\cdot}9121-10$
$= --1+9{\cdot}9121$
$= \bar{1}{\cdot}9121$

$\therefore \theta = 35° 14'.$

(*iii*) $L \tan \theta = 10+\log \tan \theta = 9{\cdot}9422$
$\therefore \log \tan \theta = 9{\cdot}9422-10$
$= -1+{\cdot}9422$
$= \bar{1}{\cdot}9422$

$\therefore \theta = 41° 12'.$

Note. In order to get logarithm of a T-ratio, 10 should be subtracted from the characteristic of the tabular logarithm.

18. The Principle of Proportional Parts. If we have to find the logarithm of a number, not contained in the tables, or of a number which lies between two numbers whose logarithms are known, we apply the Principle of Proportional Parts. It states that *the increase in the lagarithm of a number is proportional to the increase in the number itself i. e. the change in the T-ratio or in the lagarithm of a T-ratio is proportional to the change in the angle itself.*

Note. The increase or change referred to above must be **small** as compared with the number, otherwise this principle does not hold.

Ex. 1. Given $\log 37{\cdot}25 = 1{\cdot}5711$
and $\log 37{\cdot}26 = 1{\cdot}5712$,
find $\log 37{\cdot}255$.

Difference between the two numbers $= {\cdot}01$

" " their logarithms $= {\cdot}0001$

$\therefore$ for a difference of ${\cdot}01$ in the numbers difference in logarithms $= {\cdot}0001$

$\therefore$ for a difference ${\cdot}005$ in the numbers, difference in

$$\text{logarithms} = \frac{.0001}{{\cdot}01} \times {\cdot}005 = {\cdot}00005$$

$\therefore \log 37{\cdot}225 = 1{\cdot}57115.$

Ex. 2. Given that $\log \tan 36^\circ\, 47' = \bar{1}.8737$
and $\log \tan 36^\circ\, 48' = \bar{1}{\cdot}8740$
find $\log \tan 36^\circ\, 47'\, 40''$

Difference in angles $= 1' = 60''$

Difference in logarithms $= {\cdot}0003$

$\therefore$ for a difference of $60''$ in the angle difference in logarithms $= {\cdot}0003$

$\therefore$ for difference of $40''$ in angle, difference in logarithms

$$= \frac{{\cdot}0003}{60} \times 40 = {\cdot}0002$$

Hence log tan $36° \, 47' \, 40'' = \bar{1}.8737 + .0002 = \bar{1}.8739$.

Note 1. Conversely : given log tan $\theta = \bar{1}\cdot8739$ we can find that $\theta = 36° \, 47' \, 40''$.

Note. 2: Working with the four figure tables we can only aim at finding angles correct to the nearest minute. *The principle will be used only when so required.*

Exercise 25.

1. Find from the tables the values of :—
   (*i*) sin $52° \, 17'$ (*ii*) cos $32° \, 47'$ (*iii*) tan $108° \, 52'$
   (*v*) cot $56° \, 42'$
2. (*i*) log sin $43° \, 42'$ (*ii*) log cos $72° \, 27'$.

Find the values of θ lying between $0°$ and $90°$ when

3. (*i*) sin $\theta = \cdot2764$ (*ii*) cos $\theta = \cdot7282$
4. log tan $\theta = \bar{1}.7547$ (*ii*) log sin $\theta = \bar{1}\cdot5553$
5. Given log sin $27° \, 37' = \bar{1}\cdot6661$
   and log sin $27° \, 38' = \bar{1}\cdot6664$

find log sin $27° \, 37' \, 20''$ and log sin $27° \, 37' \, 45''$ correct to four places of decimals.

6. Given log sin $23° \, 18' = \bar{1}.5972$
   and log sin $23° \, 19' = \bar{1}.5975$.
   find θ, where log sin $\theta = \bar{1}.5974$,
7. If log $4 = \cdot602$ and log $5 = \cdot6990$ find log $4\cdot5$.
8. Find the values of (i) L sin $25° \, 36'$ (*ii*) L tan $44° \, 52'$.
9. Find a mean proportional between $\sqrt[3]{3473}$ and $\sqrt[5]{256\cdot4}$ (P. U. 1953)
10. Write down by using tables, the values of cos $255° \, 17'$ and cot $125° \, 15'$. (M. U.)
    (*ii*) Find (to the nearest minute) the angle whose tangent is $2\cdot4$ (B. U.)
11. If $a^2 + b^2 = 7ab$, prove that
    $\log [\frac{1}{3}(a+b)] = \frac{1}{2}[(\log a + \log b]$............ (P. U. 1949)
    [**Hint**. Add $2ab$ to both sides and take sq. root]
12. Find the most general value of x which must satisfy the two equations $\cos x = \frac{3}{5}$ and $\cot x = -\frac{3}{4}$. (P. U. 1954)

CHAPTER XII

Solution of Triangles

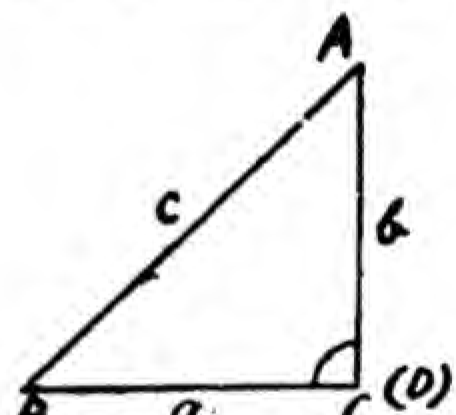
Def. The 3 sides and the 3 angles of a triangle are known as **six elements** of a triangle. Given 3 of these elements (*one at least being a side*), the other 3 can be found by calculation. The process of calculating the unknown elements from the known ones is called the **solution of a triangle.**

Problems on heights and distances in Chapter V, which the student has already tackled, were nothing but solutions of a rt. angled triangle. Here was shall solve triangles in general——(rt. angled as well as oblique-angled) making use of **logarithms** in our calculation.

1. Right-angled triangles.

Case I. Given two sides, to solve the triangle.

Ex. Given $a=321{\cdot}4$, $b=123{\cdot}9$, solve the $\triangle$. (P.U. 1952)



Sol. $\tan B = \dfrac{b}{a} = \dfrac{123{\cdot}9}{321{\cdot}4}$

$$\begin{aligned}\therefore \log \tan B &= \log 123{\cdot}9 - \log 321{\cdot}4\\ &= 2{\cdot}0931 - 2{\cdot}5073\\ &= -0{\cdot}4139 + 1 - 1\\ &= \bar{1}{\cdot}5861\\ \therefore \quad B &= 21^\circ\ 5'\\ \therefore \quad A &= 90^\circ - B = 90^\circ - 21^\circ 5'\\ &= 68^\circ\ 55'\end{aligned}$$

Again, $\sin B = \dfrac{b}{c}$

$\therefore \log \sin B = \log b - \log c.$

$$\begin{aligned}\therefore \quad \log c &= \log b - \log \sin B\\ &= \log 123{\cdot}9 - \log \sin 21^\circ\ 5'\\ &= 2{\cdot}0931 - \bar{1}{\cdot}5559\\ &= 2{\cdot}5372\\ c &= 344{\cdot}5\end{aligned}$$

$\therefore B=21^\circ 5'$, $A=68^\circ\ 55'$, $c=344{\cdot}5$

Case II. Given the hypotenuse and one side, to solve the $\triangle$.

Ex. Gizen $c=4320$, $b=2514$, solve the $\triangle$.

Sol. $\sin B = \frac{b}{c}$

$\therefore \log \sin B = \log b - \log c$

$= \log 2514 - \log 4320$

$= 3{\cdot}4000 - 3{\cdot}6355 = -{\cdot}2351 + 1 - 1$

$= \bar{1}{\cdot}7649.$

$\therefore \quad B = 35^\circ\ 41'$

$\therefore \quad A = 90^\circ - B = 90^\circ - 35^\circ 41' = 54^\circ 19'.$

Again, $\frac{a}{c} = \sin A$

$\therefore \quad \log a = \log \sin A + \log c$

$= \log \sin 54^\circ 19' + \log 4320$

$= \bar{1}{\cdot}9097 + 3{\cdot}6355$

$= 3{\cdot}5452$

$\therefore \quad a = 3510.$

Case III. Given one sidɘ and one angle, to solve the $\triangle$.

Ex. Given $b=212$, $A=15^\circ 12'$, solve the $\triangle$.

Sol. $\tan A = \frac{a}{b}$

$\therefore \quad \log a = \log \tan A + \log b$

$= \log \tan 15^\circ\ 12' + \log 212.$

$= \bar{1}{\cdot}4342 + 2{\cdot}3263$

$= 1{\cdot}7604$

$\therefore \quad a = 57{\cdot}59.$

$B = 90^\circ - A = 90^\circ - 15^\circ\ 12'$

$= 74^\circ\ 48'.$

Again $\frac{a}{c}=\sin A$

$\therefore \quad \log c=\log a-\log \sin A$

$=\log 57{\cdot}59-\log \sin 15^\circ\ 12'$

$=1{\cdot}7604-\bar{1}{\cdot}4186$

$=2{\cdot}3418$

$\therefore \quad c=219{\cdot}7.$

Case IV. Given the hypotenuse and one angle, to solve the $\triangle$.

Ex. Solve the rt. $\angle$d $\triangle$ ($C=90^\circ$), given that $c=7{\cdot}54$, $B=48^\circ\ 21'$.

Sol. $\frac{b}{c}=\sin B$

$\therefore \quad \log b=\log c+\log \sin B$

$=\log 7{\cdot}54-\log \sin 48^\circ\ 21'$

$={\cdot}8774+\bar{1}{\cdot}8734$

$={\cdot}7508$

$\therefore \quad b=5\ 633.$

$A=90^\circ-B=90^\circ-48^\circ\ 21'$

$=41^\circ\ 39'$

Again $\frac{a}{c}=\sin A$

$\therefore \quad \log a=\log c+\log \sin A$

$=7{\cdot}54+\log \sin 41^\circ\ 39'$

$={\cdot}8774+\bar{1}{\cdot}1817$

$={\cdot}6991.$

$\therefore \quad a=5.001.$

EXERCISE 26.

In the following triangles, $C=90^\circ$, solve the triangles.

1. $a=31\ 3$, $b=26{\cdot}9$.
2. $c=823{\cdot}1$, $a=237{\cdot}5$.
3. $a=1{\cdot}732$, $A=7^\circ\ 43'$.

4. $A=33° 22'$, $c=29{\cdot}9$.

5. $a=117\ 24$, $b=236{\cdot}28$, find c

6. $c=13\ 5$, $b=9{\cdot}72$.

7. $b=6{\cdot}36$, $A=38° 52'$.

8. $c=27$, $B=64° 30'$.

2. Oblique-angled triangle.

Four standard cases arise.

Case I. To solve the triangle, given its three sides.

First Method. If lengths of the sides are small (*i. e.* whole numbers of one or two digits only), the cosines of any two of the angles may be found by the cosine formulae

$$\cos A = \frac{b^2+c^2-a^2}{2bc}, \ldots\ldots\ldots\ldots$$

The third angle can then be obtained from the relation $A+B+C=180°$.

Second Method. When the lengths of the sides are bigger numbers of 3 or 4 digits, logarithms will shorten the work The best formulae for logarithmic calculation are

$\tan \dfrac{A}{2} = \sqrt{\dfrac{(s-b)(s-c)}{s(s-a)}}$, etc., when all the angles are required.

Taking logarithms, $\log \tan \dfrac{A}{2} = \frac{1}{2}\,[\log (s-b) + \log (s-c) - \log s$
$- \log (s-a)]$

This will give the value of $\dfrac{A}{2}$ and so of A. Similarly B can be found. The formulae $\sin \dfrac{A}{2} = \sqrt{\dfrac{(s-b)(s-c)}{bc}}$ and $\cos \dfrac{A}{2} = \sqrt{\dfrac{s(s-a)}{bc}}$ can also be used to find the angles but then *six* logarithms will have to be looked up instead of *four* (*i. e* s, $s-a$, $s-b$, $s-c$) as in the tangent formula.

Ex. 1. Solve the triangle, given $a=13$, $b=14$, $c=15$. (M. U.)

The sides here are small, so we apply the Cosine Formula.

$$\because \cos A = \frac{b^2+c^2-a^2}{2bc} = \frac{196+225-169}{420} = \frac{252}{420} = \cdot 6$$

$\therefore$ A $=53°\ 8'$ nearly. (from Natural Cosine tables)

$$\text{Again, } \cos B = \frac{c^2+a^2-b^2}{2ca} = \frac{225+169-196}{390} = \frac{198}{390} = \cdot 5507$$

$\therefore$ B $=59°\ 29'$ nearly.

Hence C $=180°-(A+B)=67°\ 23'$,

Note. We can also solve the triangle by the logarithmic method as illustrated in the next example, The student is advised to solve it himself by this method which is a general method.

Ex. 2. Solve the triangle, given $a=24{\cdot}76$, $b=16{\cdot}38$, $c=15{\cdot}12$,

Sol.

$$\begin{array}{lll} a=24{\cdot}76 & \therefore\ s\quad a=3{\cdot}37 & \therefore\ \log 3{\cdot}37=0{\cdot}5276 \\ b=16{\cdot}38 & s-b=11{\cdot}75 & \log 11{\cdot}75=1\ 0701 \\ \underline{c=15{\cdot}82} & \underline{s-c=13\ 01} & \log 130{\cdot}1=1{\cdot}1142 \\ 2s=65.26 & s\quad =28{\cdot}13 & \log 28{\cdot}13=1{\cdot}4492 \end{array}$$

$\therefore \quad s=28.13.$

$$\text{Now } \tan\frac{A}{2} = \sqrt{\frac{s-b)(s-c)}{s(s-a)}}.$$

$$\therefore \log \text{than} \frac{A}{2} = \tfrac{1}{2}[\log(s-b)+\log(s-c)-\log s-\log(s-a)]$$

$$=\tfrac{1}{2}[1{\cdot}0701+1{\cdot}1142-1{\cdot}4492-0{\cdot}5276]$$

$$=\tfrac{1}{2}[2{\cdot}1843-1{\cdot}9768]=\cdot 10375$$

$=\cdot 1038$ correct to 4 places of decimals.

$$\therefore \quad \frac{A}{2}=51°\ 47' \text{ or } A=103°\ 34'.$$

Again $\tan\frac{B}{2}=\sqrt{\frac{(s-c)(s-a)}{s(s-b)}}$.

$\therefore \log\ \tan\frac{B}{2}=\frac{1}{2}[\log(s-c)+\log(s-a)-\log s-\log(s-b)]$

$=\frac{1}{2}[1{\cdot}1142+0{\cdot}5276-{\cdot}4492-1{\cdot}0701]$

$=\frac{1}{2}[1{\cdot}6418-2{\cdot}5193]=-{\cdot}4375$

$=\bar{1}{\cdot}56125=\bar{1}{\cdot}5613$ correct to 4 decimal places.

$\therefore\ \frac{B}{2}=20^\circ$, or $B=40^\circ$ nearly.

Hence $C=180^\circ-(A+B)=36^\circ\ 26'$.

Note 1. It is useful to check the values of $s-a$, $s-b$, $s-c$ by adding them up as shown above :

for $(s-a)+(s-b)+(s-c)=3s-(a+b+c)$
$=3s-2s=s$,

Note 2. If greater accuracy is desired, we use the principle of proportional parts to find B.

Since $\log\tan 20^\circ=\bar{1}.5711$
and $\log\tan 20^\circ\ 1'=\bar{1}.5615$
$\therefore$ By the principle of proportional parts
$\frac{B}{2}=20^\circ 30''$ or $B=40^\circ\ 1'$.

$\therefore\ C=180-(A+B)=36^\circ\ 25'$.

Exercise 27.

Solve the triangle, given that

1. $a=8$, $b=9$, $c=10$.
2. $a=7$, $b=4\sqrt{3}$, $c=\sqrt{13}$.
3. $a=45{\cdot}73$, $b=23{\cdot}17$, $c=40{\cdot}52$. (P. U. 1952)
4. $a=345{\cdot}6$, $b=456{\cdot}6$, $c=567{\cdot}8$. (M. U.)

5. $a=32$, $b=40$, $c=66$. Find C. (P. U. 1946)

6. $a=4584$, $b=5140$, $c=3624$. (P. U. 1944)

7. Find the greatest angle, when

(*i*) the sides of a triangle are 16, 20, 33 ft. (P.U.1939)

(*ii*) the sides of a triangle are 40, 21 and 23 ft. (J. & K. U. 1955)

8. Find the greatest angle in a triangle whose sides are 7, 8 and 9 ft, having given $\log 3 = \cdot 4771213$, $\log 1\cdot 4 = \cdot 146128$, $L \cos 36° \ 42' = 9\cdot 9040529$ and difference for $60'' = \cdot 0000942$. (J. & K. U. 1949)

9. Given $a=31\cdot 9$, $b=56\cdot 31$, $c=40\cdot 27$, find the angles of the triangle as accurately as you can. (P. U.)

10. The sides of a triangle are in the ratio 4 : 5 : 6, show that one angle is twice another. (M. U.)

[**Hint.** Find the greatest and the least angles].

11. Find the greatest angle of a triangle whose sides are 2, 3, 4 having given $\log 2 = \cdot 30103$, $\log 3 = \cdot 4771213$, $L \tan 52° \ 14' = 10\cdot 1108195$, $L \tan 52° \ 15' = 10\cdot 111100$. (D. U. qualifying)

12. Given $a=229\cdot 2$, $b=181\cdot 2$, $c=257$, solve the triangle. (P. U. 1955)

Case II. To solve the triangle, given two sides and the included angle.

Let the sides a, b and the angle C be given. The formula suitable for the use of logarithms is the Napier's Analogy

$$i.e., \ \tan \frac{A-B}{2} = \frac{a-b}{a+b} \cot \frac{C}{2} = \frac{a-b}{a+b} \tan \left(90° - \frac{C}{2}\right),$$

if $a>b$ ...(1)

[But if $a<b$ it should be used in the form

$$\tan \frac{B-A}{2} = \frac{b-a}{b+a} \cot \frac{C}{2} \ ]$$

Taking logs of both sides of (1) we get,

$$\log \tan\frac{A-B}{2}=\log (a-b)-\log (a+b)+\log \tan\left(90^\circ-\frac{C}{2}\right)$$

From this formula $\frac{A-B}{2}$ can be obtained. As C is known we can get $\frac{A+B}{2}$ from the relation $\frac{A+B}{2}=\frac{180^\circ-C}{2}=$ $90^\circ-\frac{C}{2}$.

From these two, adding and subtracting, A and B can be found.

The 3rd side c can then be determined from the Sine formula $\frac{c}{\sin C}=\frac{a}{\sin A}$ which gives $c=\frac{a \sin C}{\sin A}$. After taking logs of both sides, c can be found from $\log c=\log a+\log \sin C -\log \sin A$ with the help of log tables.

Thus the triangle is completely solved.

Note 1. If a and b are small integral numbers, c can also be found easily from $c^2=a^2+b^2-2ab \cos C$.

Note 2. If a, b and A—B are given, C can be found by the formula $\tan\frac{A-B}{2}=\frac{a-b}{a+b}\cot\frac{C}{2}$.

Ex. 1. Solve the triangle given $a=36{\cdot}21$, $c=30{\cdot}14$, $B=78^\circ\ 10'$.

Here $a>c$.

using $\tan\frac{A-C}{2}=\frac{a-c}{a+c}\cot\frac{B}{2}$ or $\frac{a-c}{a+c}\tan\left(90^\circ-\frac{B}{2}\right)$

we get, $\tan\frac{A-C}{2}=\frac{6{\cdot}07}{66{\cdot}35}\tan 50^\circ\ 55'$

$$\therefore \log\tan\frac{A-C}{2}={\cdot}7832+{\cdot}0940-1{\cdot}8218.$$
$$=\bar{1}{\cdot}0518$$

$\therefore \dfrac{A-C}{2}=6^\circ\ 26'$

But $\dfrac{A+C}{2}=90^\circ-\dfrac{B}{2}=50^\circ\ 55'$

$\therefore$ Adding, $A=57^\circ\ 21'$ and subtracting, $C=44^\circ\ 29'$.

Now from $\dfrac{a}{\sin A}=\dfrac{b}{\sin B}$ we have $b=\dfrac{a \sin B}{\sin A}$

$$\therefore \log b=\log a+\log \sin B-\log \sin A$$
$$=\log 36{\cdot}21+\log \sin 68^\circ\ 10'-\log \sin 57^\circ\ 21'$$
$$=1{\cdot}5588+\bar{1}.9907+\bar{1}.9253=1{\cdot}6242$$
$$\therefore \quad b=42{\cdot}02.$$

Ex 2. If $b=7$, $c=3$ and $\angle A=60^\circ$, find the angles B and C of the triangle, given that

$\log 2={\cdot}3010$, $\log 3={\cdot}4771$, L tan $34^\circ\ 42'=9{\cdot}8404$.
difference for $1'=.0003$.

$$\tan\frac{B-C}{2}=\frac{b-c}{b+c}\cot\frac{A}{2}$$
$$=\frac{4}{10}\cot 30^\circ=\frac{4}{10}\sqrt{3}.$$

$$\therefore \log\tan\frac{B-C}{2}=\log 4+\log\sqrt{3}-\log 10$$
$$=2\log 2+\tfrac{1}{2}\log 3-1$$
$$={\cdot}6020+{\cdot}2386-1$$
$$=\bar{1}{\cdot}8406.$$

$$\therefore \text{L}\tan\frac{B-C}{2}=10+\bar{1}{\cdot}8406=9{\cdot}8406$$

which is greater by ${\cdot}0002$ than L tan $34^\circ\ 42'$.

But difference for $1'$ (*i. e.* $60''$) $={\cdot}0003$

$\therefore$ difference of ${\cdot}0002$ is due to $\frac{2}{3}\times 60''=40''$

Hence $\dfrac{B-C}{2}=34^\circ\ 42'\ 40''$

and $\frac{B+C}{2}=90^\circ-\frac{A}{2}=60^\circ$

$\therefore$ Adding, $B=94^\circ\ 42'\ 40''$

and subtracting, $C=25^\circ\ 17'\ 20''$.

EXERCISE 28.

Solve the triangle, given that

1. $b=\sqrt{3}, c=1, A=30^\circ$ (J. & K. U. 1960)
2. $a=\sqrt{3}+1, b=\sqrt{3}-1, C=60^\circ$.
3. $b=8, c=5, A=36^\circ\ 52'$. (P. U. 1936)
4. $b=25{\cdot}1, c=14{\cdot}7$ and $A=47^\circ$. (P. U. 1948)
5. $a=21{\cdot}35, b=35{\cdot}21$ and $C=50^\circ\ 48'$. (P. U. 1953)

6. Given that $b=130$, $c=72$, $A=42^\circ$, find the other two angles of the triangle. (P. U. 1949 S)

7. Two sides of a triangle are 80 ft and 100 ft. and the included angle is 60°. Find the other two angles having given $\log 3={\cdot}47712$ and $L \tan 10^\circ\ 53'\ 36''=9{\cdot}28432$. (J. & K.U. 1950)

8. Two sides of a triangle are 5 and 4 yards and the included angle is 60°. Find the other angles having given $\log 3={\cdot}47712$, $L \tan 10^\circ\ 53'=9{\cdot}28390$ and $L \tan 10^\circ\ 54'=9{\cdot}28458$. (J. & K. U. 1951)

9. Two sides of a triangle are in the ratio 16 : 9 and the included angle is $102^\circ\ 48'$. Find the other angles.

$$\left[\text{Hint. } \frac{b}{c}=\frac{16}{9} \quad \therefore \frac{b-c}{b+c}=\frac{7}{25}\right]$$

10. Two sides of a triangle are 3 and 5, and the included angle is 75°. Find the other angles, having given : $\log 2={\cdot}3010$; $\log \tan 52^\circ\ 30'={\cdot}1150$, $\log \tan 18^\circ=\bar{1}{\cdot}5118$, diff. for $3'={\cdot}0013$.

11. Find all the angles, correct to the nearest second, of the triangle, in which $b=16{\cdot}58$, $c=50{\cdot}2$, $A=37^\circ$. (J. & K. U. 1954)

12. Given $b=68$, $c=27$, $B-C=70^\circ$, solve the $\triangle$.

[**Hint.** From $\tan\dfrac{B-C}{2}=\dfrac{b-c}{b+c}\cot\dfrac{A}{2}$, we get

$$\tan\frac{A}{2}=\frac{b-c}{b+c}\cot\frac{B-C}{2}]$$

Using logs we get A. But B+C1=80°−A etc.]

Case III. To solve the triangle, given one side and any two angles.

Suppose A, B and c are given.

C can be got from the relation $A+B+C=180°$.

From $\dfrac{a}{\sin A}=\dfrac{b}{\sin B}=\dfrac{c}{\sin C}$, the values of a and b are calculated from the relations :—

$a=\dfrac{c\sin A}{\sin C}$ and $b=\dfrac{c\sin B}{\sin C}$ with the help of logarithmic tables.

Ex. Solve the $\triangle$ ABC, given

$$A=35°\ 17',\ C=45°\ 13',\ b=42{\cdot}1.$$

Sol $B=180°-(A+C)=99°\ 30'$.

$\because \dfrac{a}{\sin A}=\dfrac{b}{\sin B}$, $\therefore a=\dfrac{b\sin A}{\sin B}$

$\therefore \log a=\log b+\log\sin A-\log\sin B$

$=\log 42{\cdot}1+\log\sin 35°\ 17'-\log\sin 99°\ 30'$

$=1{\cdot}6243+\bar{1}{\cdot}7616-\bar{1}{\cdot}9940$

$=1{\cdot}3919$

$\therefore\quad a=\text{anti-log } 1{\cdot}3919=24{\cdot}65.$

Again from $\dfrac{c}{\sin C}=\dfrac{b}{\sin B}$ we get $c\ \dfrac{b\sin C}{\sin B}$

$\therefore \log c=\log b+\log\sin C-\log\sin B$

$=\log 42{\cdot}1+\log\sin 45°\ 13'-\log\sin 99°\ 30'$

$=1{\cdot}6243+\bar{1}{\cdot}8511-\bar{1}{\cdot}9940=1{\cdot}4814$

$\therefore\quad c=\text{anti-log } 1{\cdot}4814=3030,$

Exercise 29.

Solve the $\triangle$, given that

1. $A=80°$, $B=53°$, $a=152$. (P. U. 1932)
2. $B=83° 36'$, $C=31° 54'$ $a=53$ inches (P. U. 1935)
3. $B=64° 23'$, $C=72° 43'$, $a=18{\cdot}92$. (P. U. 1950)
4. $a=15{\cdot}72$ ft., $A=41° 30'$, $B=72° 45'$. (P. U. 1953)
5. $A=72° 43'$, $B=64° 23'$, $C=473$. (P. U. 1955 S)
6. In a triangle, base $=7$ and base angles are $129° 23'$ and $38° 36'$. Find the length of the shorter side.
7. A and B are two points 50 ft. apart on the same bank of a canal. C is a point on the opposite bank such that the angles CAB and CBA are $22° 30'$ and $112° 30'$. Show that the width of the canal is 25 ft.

Case IV. To solve the triangle given two sides and the angle opposite to one of them.

Before giving the method of logarithmic solution we give here the geometrical discussion of the case.

Let a, b and A be given. In the figure draw $\angle CAX=A$ and cut off $AC=b$. Draw CD perpendicular to AX. Then $CD=b \sin A$. With centre C and radius a draw an arc. The following possibilities will arise :—

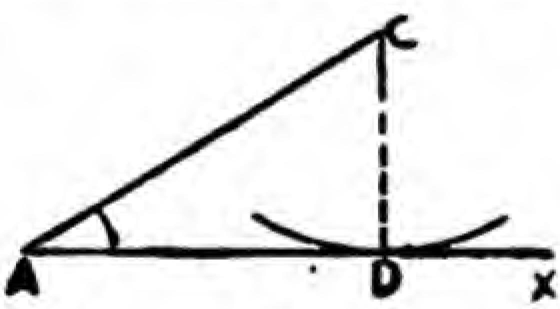


Fig 1.

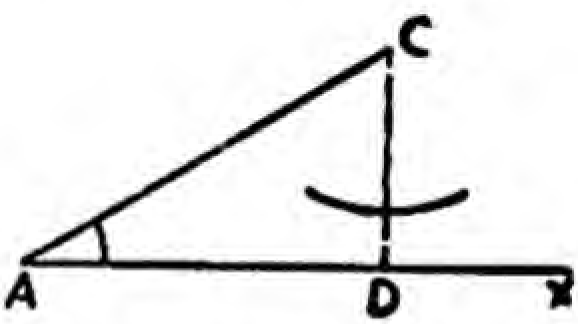


Fig 2

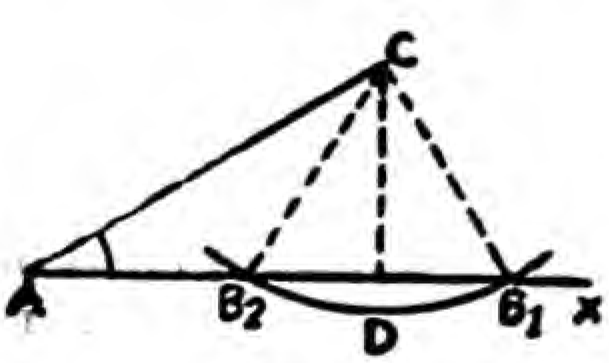


Fig 3.

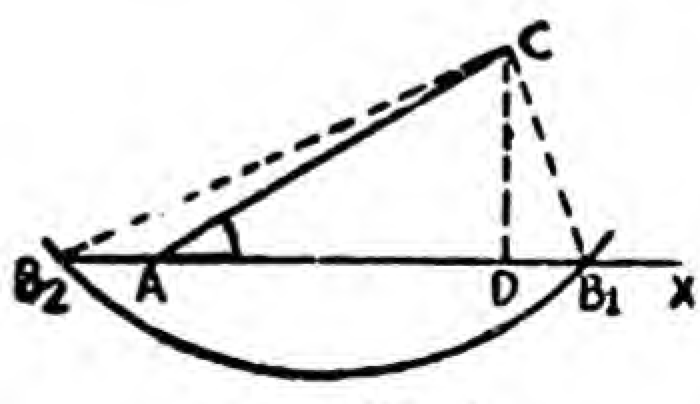


Fig 4.

1. When A is acute.

(*i*) If $a <$ CD ($= b \sin A$) the arc does not cut AX as in Fig. 1. That is, no triangle is formed.

(*ii*) If $a =$ CD ($= b \sin A$) the arc touches AX at D. Therefore one triangle (rt. $\angle$ed) *viz.* ACD is formed as in Fig. 2.

(*iii*) If $a >$ CD($= b \sin A$) the arc cuts AX in two points B_1 and B_2. These points are on the same side of A if $a < b$ as in Fig. 3. Then two triangles ACB_1 and ACB_2 are formed satisfying the given elements a, b and A. But these points are on opposite sides of A if $a > b$. In that case one triangle viz. ACB_1 alone is formed which satisfies the given elements a, b and A as in fig 4.

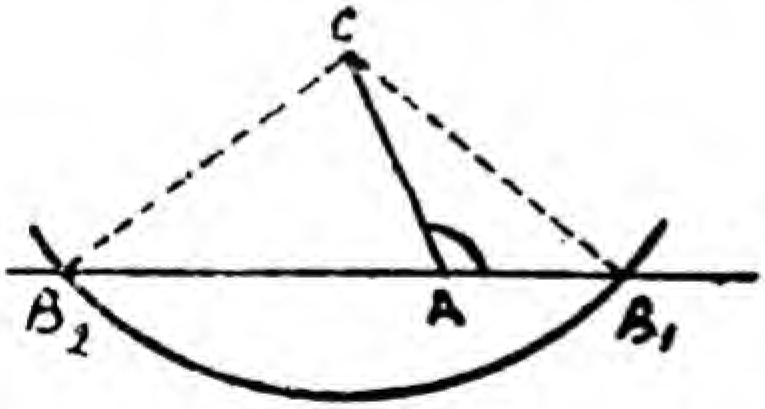
(2) When $A = 90°$

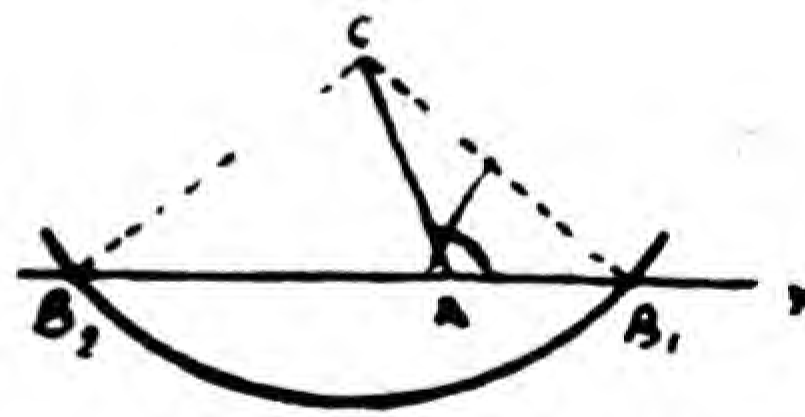
(*i*) If $a > b$, the arc will cut AX in two points on opposite sides of A forming two equal rt. $\angle$ed triangles and therefore only one triangle is formed.

(*ii*) If $a < b$, the arc will not cut AX and therefore no triangle is formed

(*iii*) If $a = b$, the arc will touch AX at A and therefore, no $\triangle$ is formed.

(3) When A is obtuse.





(*i*) If $a > b$, the arc cuts AX in two points on opposite sides of A but only the $\triangle$ ACB_1 satisfies the given elements.

(*ii*) If $a < b$, the arc either cuts AX at two points or touches it on the left of A or does not cut it at all. In either case no triangle is formed satisfying the given elements.

(*iii*) If $a=b$, the arc cuts AX at only one poin on the left of A. Thus, again no $\triangle$ is formed with the given elements.

From the above we conclude that when two sides a, b and an angle, say A, opposite to one of them is given it is possible to construct two different triangles provided :—

(*i*) A is acute, and

(*ii*) $a>b \sin A$ but $<b$. That is when a lies between $b \sin A$ and b.

This is called the **ambiguous case.**

Note. 1. The case of two distinct solutions arises only when the given angle is opposite to the shorter of the two given sides.

Note. 2. If there are two triangles as in fig. 3. the angles AB_1C and AB_2C are supplementary ; for then in the insosceles $\triangle\ CB_1B_2$,

$$\angle AB_2C = 180° - \angle CB_2B_1 = 180° - \angle CB_1B_2$$

The method of solution :—

To solve the triangle given a, b, and A.

From $\dfrac{b}{\sin B} = \dfrac{a}{\sin A}$ we have $\sin B = \dfrac{b \sin A}{a}$

$$\therefore \log \sin B = \log b + \log \sin A - \log a \ldots\ldots\ldots (1)$$

Now three cases arise according as the value of log sin B obtained from (1) is positive, zero or negative.

(*i*) If log sin B is positive then $\sin B > 1$, which is impossible. Hence there is no solution.

(*ii*) If $\log \sin B = 0$, then $\sin B = 1$ *i. e.* $B = 90°$. Hence there is one solution and the triangle is rt. angled.

(*iii*) If log sin B is negative, $\sin B < 1$, which gives two values of B namely $B_1 < 90°$ (*i. e.* acute) and B_2 the supplement of B_1 (*i. e.* obtuse). Of the two values of B, that value is to be *rejected* which when added to the given angle A makes the sum $\geqslant 180°$. Thus. there is either *one* solution or *two* solutions according as only one value of B or both the values of B are admissible.

The third angle C can then be found from the relation $C=180°-(A+B)$ C will have two values C_1, C_2 if B has two values B_1, B_2.

The remaining side c is then determined from $\frac{c}{\sin C}=\frac{a}{\sin A}$

i. e. from $c=\frac{a \sin C}{\sin A}$, with the help of log-tables.

c will also have two values c_1, c_2 corresponding to the two values of C

Ex. 1. Solve the triangle, given $b=7$, $c=10$, $\angle B=51°$ (P. U. 1937)

From $\frac{c}{\sin C}=\frac{b}{\sin B}$ we have $\sin C=\frac{c \sin B}{b}$

$\therefore$ $\log \sin C=\log c+\log \sin B-\log b$

$=\log 10+\log \sin 51°-\log 7$

$=1+\bar{1}.8905-{\cdot}8451$

$={\cdot}0454$, a positive quantity.

$\therefore$ $\sin C>1$. But this is impossible.

Hence there is no solution.

Note. The student should satisfy himself that tables give no angle corresponding to this value.

Ex. 2. Solve the triangle, given $a=246{\cdot}7$ $b=342{\cdot}5$, $B=32°\ 17'$.

Here we note that the side opposite to the given angle *i. e.*, $b>a$.

$\therefore$ We will have only one solution.

From $\frac{a}{\sin A}=\frac{b}{\sin B}$ we have $\sin A=\frac{a \sin B}{b}$

$\therefore$ $\log A=\log a+\log \sin B-\log b$

$=\log 246{\cdot}7+\log \sin 32°\ 17'-\log 342{\cdot}5$

$=2{\cdot}3921+\bar{1}.7276-2{\cdot}5346$

$=\bar{1}.5851$

$\therefore$ $A = 22° 37'$ or its supplement $157° 23,$

The second value of A is inadmissible because it makes the sum $A+B > 180°$.

$\therefore C = 180° - (A+B) = 125° 6'$.

Now from $\dfrac{c}{\sin C} = \dfrac{b}{\sin B}$ we have $c = \dfrac{b \sin C}{\sin B}$

$$\begin{aligned}\therefore \log c &= \log b + \log \sin C - \log \sin B \\ &= \log 342{\cdot}5 + \log \sin 125° 6' - \log \sin 32° 17' \\ &= \log 342{\cdot}5 + \log \sin 54° 54' - \log \sin 32° 17' \\ &= 2{\cdot}5346 + \bar{1}{\cdot}9128 - \bar{1}{\cdot}7276 \\ &= 2{\cdot}7198\end{aligned}$$

$\therefore$ $c = 524{\cdot}6$ (from anti-log tables)

Ex. 3. Solve the triangle, given $a = 4{\cdot}7$, $c = 1{\cdot}3$, $C = 15°$ (M.U.)

Here $c < a$, hence there is possibility of two solutions.

From $\dfrac{a}{\sin A} = \dfrac{c}{\sin C}$ we get $\sin A = \dfrac{a \sin C}{c} = 4{\cdot}7 \times \dfrac{\sin 15°}{1{\cdot}3}$

$$\begin{aligned}\therefore \log \sin A &= \log 4{\cdot}7 + \log \sin 15° - \log 1{\cdot}3 \\ &= {\cdot}6721 + \bar{1}{\cdot}4130 - {\cdot}1139 \\ &= \bar{1}{\cdot}9712\end{aligned}$$

$\therefore A = 69° 22'$ or its supplement $110° 38'$

Both the values of A are admissible because the sum of the obtuse value of A and the given value of C is $< 180°$.

Let the acute value of A be denoted by A_1 and the obtuse value by A_2.

$\therefore B_1 = 180° - (A_1 + C) = 95° 38'$.

and $B_2 = 180° - (A_2 + C) = 54° 22'$.

Now there will be two values of b, say b_1 and b_2.

From $\dfrac{b_1}{\sin B_1} = \dfrac{c}{\sin C}$ we have $b_1 = \dfrac{c \sin B_1}{\sin C} = \dfrac{1{\cdot}3 \times \sin 95° 38'}{\sin 15°}$

$$\begin{aligned}\therefore \log b_1 &= \log 1{\cdot}3 + \log \sin 95° 38' - \log \sin 15° \\ &= {\cdot}1139 + \bar{1}{\cdot}9978 - \bar{1}{\cdot}4130 \\ &= {\cdot}6987\end{aligned}$$

$\therefore \quad b_1 = 4{\cdot}997$

Similarly, $b_2 = \dfrac{c \sin B_2}{\sin C} = \dfrac{1{\cdot}3 \times \sin 54°\, 22'}{\sin 15°}$

$$\begin{aligned} \log b_2 &= \log 1{\cdot}3 + \log \sin 54°\, 22' - \log \sin 15° \\ &= {\cdot}1139 + \overline{1}{\cdot}9100 - \overline{1}{\cdot}4130 \\ &= {\cdot}6109 \end{aligned}$$

$\therefore \; b_2 = 4{\cdot}083.$

Hence the two solutions are :—

1. $A_1 = 69°\, 22'$, $B_1 = 95°\, 38'$, $b_1 = 4{\cdot}997$
2. $A_2 = 110°\, 42'$, $B_2 = 54°\, 22'$, $b_2 = 4{\cdot}083$

Note. It is not sufficient to say that for two solutions the side opposite to the angle should be less than the other. The complete condition is : $b \sin A < a < b$.

EXERCISE 30.

Solve the triangle, given that

1. $b = 16$, $c = 25$, $B = 33°\, 15'$ (P. U. 1948)
2. $b = 6{\cdot}5$, $c = 3{\cdot}3$, $C = 30°\, 31'$. (M.U.)
3. $a = 11$, $b = 17$, $A = 30°\, 21'$. (P. U. 1937)
4. $c = 421{\cdot}9$, $a = 531{\cdot}4$, $A = 70°\, 15'$. (P. U. 1942 S)
5. $a = 182{\cdot}5$, $b = 82{\cdot}5$, $A = 72°\, 15'$. (J. & K. U. 1957)
6. $a = 8231$, $c = 7295$, $C = 42°\, 27'$ (P. U. 1956)
7. Find the other angles of a triangle when one angle is $112°\, 4'$, the side opposite to it is 573 ft. and another side is 394 ft., given that :—

   $\log 5{\cdot}73 = 0{\cdot}7581546$, $\log 3{\cdot}94 = 0{\cdot}5954962$

   $L \cos 22°\, 4' = 9{\cdot}9669614$, $L \sin 39°\, 35' = 9{\cdot}8042757$,

   $L \sin 39°\, 36' = 9{\cdot}8044284$ (A. U.)
8. Point out whether the solutions of the following triangles are ambiguous or not.

   (*i*) $A = 30°$, $c = 250$ and $a = 125$ ft.

   (*ii*) $A = 30°$, $c = 150$, $a = 200$ ft.

9. $a=2$, $b=3$, $A=30°$. Find the other angles, given that $\log 2=\cdot 30103$, $\log 3=\cdot 47712$

$L \sin 48° 35'=9\cdot 87501$, $L \sin 48° 36'=9\cdot 87513$.

(J. & K. U. 1952)

10. If $A=50°$, $b=1071$, $a=873$, find to the nearest second, angle B. Given $\log 1\cdot 071=\cdot 029789$

$L \sin 70°=9\cdot 972986$, $\log 8\cdot 73=\cdot 941014$
$L \sin 70° 1'=9\cdot 973032$. (J. & K. U. 1958)

3. Trigonometrical discussion of the Ambiguous case

From the geometrical discussion given already we have seen that given two sides and the angle opposite to one of them, sometimes *two* triangles are possible, sometimes *one* and sometimes *none*. Now we shall discuss the problem from the point of view of Trigonometry.

Let a, b and A be given.

From $\dfrac{b}{\sin B}=\dfrac{a}{\sin A}$, we have $\sin B=\dfrac{b \sin A}{a}$

(1) First let A be acute.

Three cases arise :—

Case (*i*) If $a<b \sin A$, then $\sin B>1$, which is impossible.

$\therefore$ there is no solution.

Case (*ii*) $a=b \sin A$, then $\sin B=1$, $\therefore B=90°$

$\therefore$ there is only one solution and the triangle is a rt. angled one.

Case (*iii*) If $a>b \sin A$, then $\sin B<1$

$\therefore$ B has two values (one acute and the other obtuse) which are supplementary.

Both of them are not always admissible.

Now three sub-cases arise :—

(*a*) If $a>b$, then $A>B$, *i. e.* $B<A$.

$\therefore$ B is acute (as A is given to be acute). Hence the obtuse value of B is inadmissible.

$\therefore$ there is only one solution.

(*b*) If $a=b$, then $A=B$.

$\therefore$ B is also acute. Hence the obtuse value of B is inadmissible.

$\therefore$ there is only one solution and the triangle is isosceles.

(*c*) If $a<b$, then $A<B$ *i. e.* $B>A$.

$\therefore$ B may be either acute or obtuse and hence both the values of B are admissible.

$\therefore$ there are two solutions.

This, in fact, is known as the **Ambiguons Case.**

(2) Secondly, let $A=90°$

Here $\text{Sin } B = \dfrac{b}{a}$

Since a triangle cannot have more than one right angle therefore we cannot have more than one solution. But when $a<b$ or $=b$, $\sin B \geqslant 1$; hence no triangle is possible.

3. Thirdly, let A be obtuse.

Three cases arise :—

Cases (*i*) and (*ii*),. If $a<b$ or $=b$, then $A \leqslant B$ *i. e*; $B \geqslant A$

Since A is obtuse therefore B must be obtuse. But this is impossible as a triangle cannot have two obtuse angles.

$\therefore$ there is no solution.

Case (*iii*), If $a>b$, $A>B$ *i. e*, $B<A$.

Hence B may be either acute or obtuse. But obtuse value will be impossible hence only acute value of B is admissible.

$\therefore$ there is only one solution.

From the foregoing discussion we see that the only case in which an **ambiguous solution** can arise, if at all, is when the smaller of the two given sides is opposite to the given angle.

Thus given *a. b*, A, the conditions for the ambiguous case are :

(1) A is acute

(2) $b \sin A < a < b$.

4. Algebraical Discussion.

1. First, let A be acute.

$\therefore \cos A = \dfrac{b^2+c^2-a^2}{2bc}$.

$\therefore$ Writing it as a quadratic in c (which is unknown)

$$c^2-2c.\, b\cos A + b^2 - a^2 = 0 \ldots\ldots\ldots (1)$$

Solving, $c = \dfrac{2b\cos A \pm \sqrt{4b^2 \cos^2 A - 4(b^2-a^2)}}{2}$

$= b\cos A \pm \sqrt{a^2 - b^2 \sin^2 A}$

Now three cases arise :—

Case (*i*) If $a < b \sin A$, the discriminant $a^2 - b^2 \sin^2 A$ is negative

$\therefore$ the two values of c are imaginary.

$\therefore$ there is no solution.

Case (*ii*) If $a = b \sin A$, the discriminant $a^2 - b^2 \sin^2 A = 0$

$\therefore$ the values of c are equal, each $= b\cos A$, so that $\cos A = \dfrac{c}{b}$

But from the projection formula $c = a\cos B + b\cos A$.

$\therefore b\cos A = a\cos B + b\cos A$

$\therefore a\cos B = 0$ or $\cos B = 0$. Hence $B = 90°$.

$\therefore$ there is only one solution and the triangle is right angled.

Case (*iii*) If $a > b \sin A$, the discriminant $a^2 - b^2 \sin^2 A$ is positive.

$\therefore$ the two values of c are real, but not necessarily both positive.

Sum of the roots of the quadratic (1) $= 2b\cos A$ which is positive, A being acute.

Product of the roots of the quadratic (1) $= b^2 - a^2$

Three sub-cases further arise :—

(*a*) If $a<b$, the product of the roots is positive and the sum being also positive both the values of c are positive.

$\therefore$ there are two solutions.

This is the **ambiguous case.**

Hence the conditions for the ambiguous case are :—

A is acute, and $b \sin A<a<b$

(*b*) If $a=b$, the product of the roots$=0$, and the sum being positive, one value of c is **zero** and other is positive.

$\therefore$ there is only one solution (Isosceles $\triangle$).

(*c*) If $a>b$, the product of the roots is negative, and sum being positive, one value of c is negative which is meaningless.

$\therefore$ there is only one solution.

2 Secondly, Let $A=90°$

then $c=\pm\sqrt{a^2-b^2}$

Case (*i*) and (*ii*). If $a<b$, c will be imaginary. If $a=b$, $c=0$

$\therefore$ no solution is possible.

Case (*iii*) If $a>b$, c will have two values, one positive and the other negative.

The negative value is meaningless.

$\therefore$ there is only one solution.

(3) Thirdly, let A be obtuse.

Then sum of the roots of quadratic (1)$=2b \cos A$

(which will be negative)

and product of the roots of quadratic (1)$=b^2-a^2$

Now three sub-cases arise :—

Case (*i*) if $a<b$, the product is positive and the sum is negative.

$\therefore$ both the values of c are negative.

$\therefore$ there is no solution.

Case (*ii*) If $a=b$, the product$=0$ and the sum is negative.

$\therefore$ one value of $c=0$ and the other is negative.

∴ there is no solution.

Case (*iii*) If $a>b$, both the product and the sum are negative.

∴ one value of c is negative and the other is positive.

∴ there is no solution.

Exercise 31.

1. In a triangle a, b, A are given, prove geometrically that there will be two solutions if A is acute and $b>a>b \sin A$. (P. U.)

2. (*i*) If a, b, A be given parts of a triangle, and c_1, c_2 be the two values of the third side, show that :

$c_1+c_2=2b \cos A$, and $c_1 c_2=b^2-a^2$. (D. U.)

(*ii*) Show that the difference between the two values of c is $2\sqrt{a^2-b^2 \sin^2 A}$.

3 In the ambiguous case a, b, B being given where $a>b$, if c, c' be the values of third side, show that

$c^2-2cc' \cos 2B+c'^2=4b^2 \cos^2 B$. (D. U. 1934)

4. In a triangle ABC, b, c, B are given, also $b<c$; show that $(a_1-a_2)^2+(a_1+a_2)^2 \tan^2 B=4b^2$

where a_1, a_2 are the two values of the third side. (P. U)

5 State the number of solutions in the following :—

(*i*) $B=30°$, $c=16$, $b=8$

(*ii*) $B=30°$, $c=16$, $b=12$

(*iii*) $B=30°$, $c=16$, $b=6$.

CHAPTER XIII

Heights and Distances.

The most important practical use of the methods of solving triangles consists in their application to the determintion of heights and distances of inaccessible objects and is of great importance for a surveyor, an engineer or a map-maker.

Simple problems involving solutions of right-angled triangles were discussed in Chapter V. Here we shall deal with problems requiring solutions of triangles in general. Solutions of the problems will be much simplified by *drawing neat diagrams and marking the given lengths and angles.* The students should be able to write down sufficient number of equations by means of which the unknown quantities can be determined, and choose only such formulae as are suitable for logarithmic calculation.

Two types of problems will be illustrated below by examples :—

1. Those in which the objects all lie in one plane.

2. Those in which the objects do not lie in one plane.

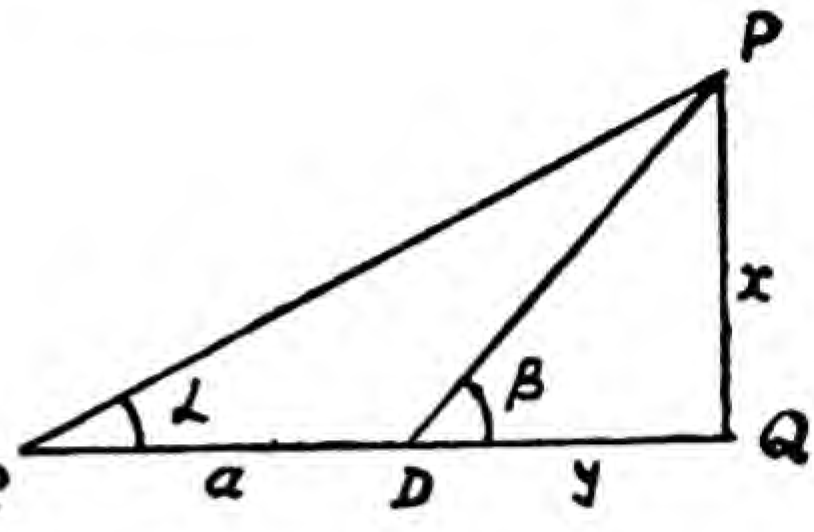
Def. For measuring angles a practical surveyor uses :—

1. **A Theodolite**—which is an instrument for measuring angles in a horizontal or a vertical plane (*i. e.* angles of elevation and depression).

2. **The Sextant**—which is an instrument for measuring angles that do not lie in a horizontal or a vertical plane (*e. g.* an angle between lines drawn from the observer's eye to two distant objects not in the same line with the observer).

1. To find the height and distance of an inaccessible object using two points of observation.

Ex. PQ is a tower, C and D are two points distance a apart in a horizontal straight line through Q the foot of the tower. If the angles of elevation of P at C and D are α and β respectively, find the height of the tower and its distance from D.



Let PQ$=x$ and DQ$=y$

In $\triangle$ PCD, we have

$$\frac{\text{PD}}{\sin\alpha}=\frac{a}{\sin \text{CPD}}\ i.e.\ \frac{a}{\sin(\beta-a)}$$

$$\therefore \text{PD}=\frac{a\sin\alpha}{\text{Sin}\ (\beta-a)}$$

But $x=$PD $\sin\beta$
and $y=$PD $\cos\beta$

$$\therefore\ x=\frac{a\sin\alpha\sin\beta}{\sin(\beta-a)} \text{ and } y=\frac{a\sin\alpha\cos\beta}{\sin(\beta-\alpha)}.$$

Thus x and y have been obtained in terms of expressions suitable for logarithmic calulation.

Ex. 2. A person walking along a straight road observes that at two consecutive mile-stones the angles of elevation of a hill in front of him are 30° and 75° Find the height of the hill. (K. U. 1955)

Let ht. of the hill$=x$

then, from Ex. 1, $x=\dfrac{a\sin\alpha\sin\beta}{\sin\beta-\alpha}=\dfrac{1.\sin 30^\circ \sin 75^\circ}{\sin 45^\circ}$ miles

$$\therefore\ \log x=\log\sin 30^\circ+\log\sin 75^\circ-\log\sin 45^\circ$$
$$=\bar{1}{\cdot}6990+\bar{1}{\cdot}9849-\bar{1}{\cdot}8495$$
$$=\bar{1}{\cdot}8344$$
$$\therefore\quad x={\cdot}683 \text{ miles.}$$

Ex. 3. If in Ex. 1, C and D are not in a horizontal line through Q, find the height of the tower, being given that $\angle$PCQ$=\alpha$, $\angle$PCD$=\beta$ and $\angle$PDC$=\gamma$.

From $\triangle$ PCD, we get

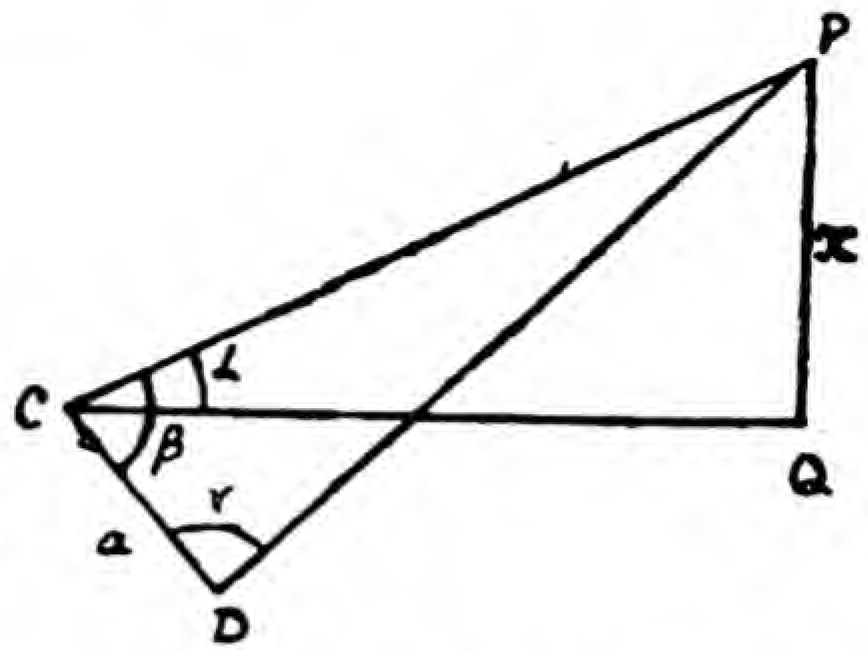
$$\frac{PC}{\sin\gamma}=\frac{a}{\sin\angle CPD}$$

$$i\ e.\ \frac{a}{\sin\left(180-\overline{\beta+\gamma}\right)}$$

$$\therefore PC=\frac{a\sin\gamma}{\sin(\beta+\gamma)}$$

Now From $\triangle$ PCQ, we have

$$x=PC\sin\alpha=\frac{a\sin\gamma\sin\alpha}{\sin(\beta+\gamma)}$$



Thus x is determined by a formula suitable for logarithmic calculation.

Ex. 4. PQ is a hill and P is its top. C and D are two points distance a apart along a straight line inclined at an angle α to the horizontal and in the same vertical plane with PQ. If the angles of elevation of P at C and D are θ and ϕ respectively, find the height of the hill.

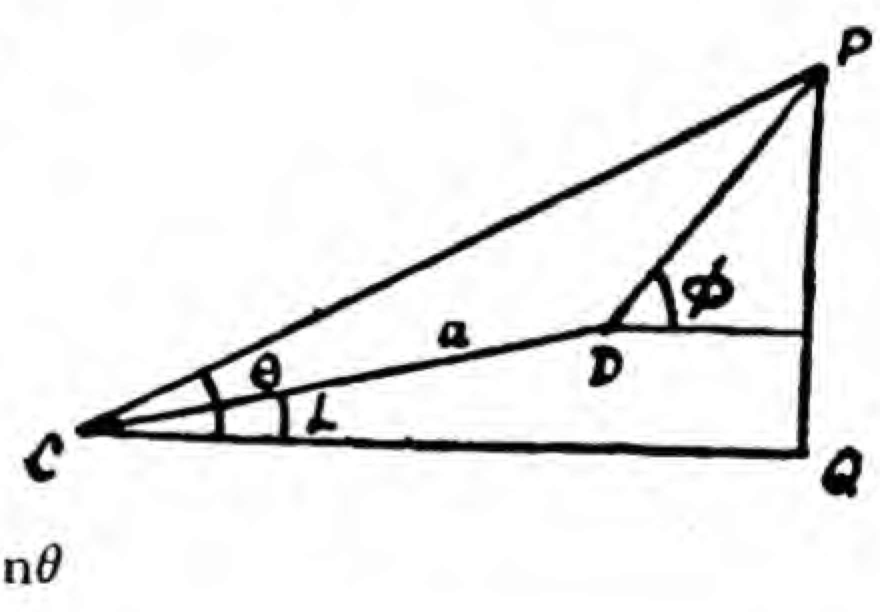
In $\triangle$PCD, $\angle PCD=\theta-\alpha$
and $\angle CPD=\phi-\theta$
$\therefore\ \angle PDC=180°-\phi+\alpha$

$$\text{Now}\ \frac{PC}{\sin(180°-\phi+\alpha)}=\frac{a}{\sin(\phi-\theta)}$$

But in $\triangle$ PCQ,

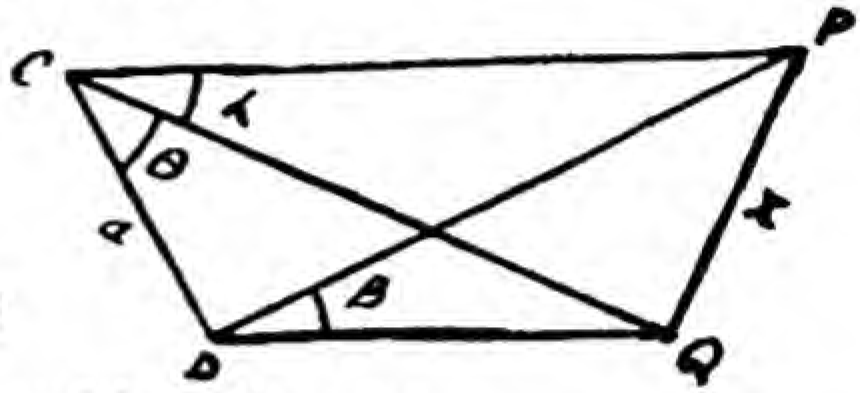
$$PQ=PC\sin\theta$$

$$PQ=\frac{a\sin(\phi-\alpha)\sin\theta}{\sin(\phi-\theta)}$$



Ex. 5. PQ is a tower, and C and D are two points distance a apart in a horizontal plane with Q, CD making an angle θ with CQ. The angles of elevation of P at C and D are α and β respectively. Find the height of the tower.

Since the lines CQ and DQ are in the horizontal plane
$\therefore$ $\triangle$s CPQ and DPQ are rt. angled.

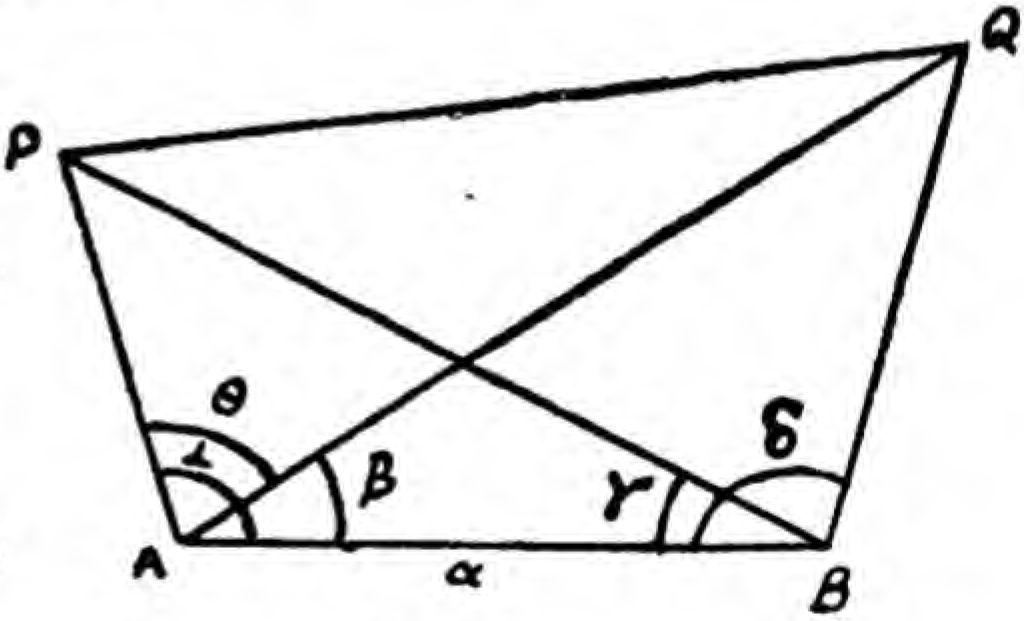


Let PQ $= x$
Then CQ $= x \cot \alpha$ and
DQ $= x \cot \beta$
In the $\triangle$QCD, we have

$$QD^2 = CQ^2 + CD^2 - 2CQ \cdot CD \cos \theta$$

or $x^2 \cot^2 \beta = x^2 \cot^2 \alpha + a^2$
$-2x\alpha \cot \alpha \cos \theta$, which gives x.

Ex. 6. Let P and Q be the objects and A and B be two accessible points, distance a apart, from which both are visible. At A the angles PAB and QAB are observed to be α and β. Also at B the angles PBA and QBA are observed to be γ and δ. If $\angle$PAQ $= \theta$, find *a* method for finding the distance PQ suitable for logarithmic calculation.



From $\triangle$ PAB

$$\frac{AP}{AB} = \frac{\sin PBA}{\sin APB}$$

i.e. $\dfrac{\sin \gamma}{\sin (180° - \alpha + \gamma)}$

$$\therefore AP = \frac{a \sin \gamma}{\sin (\alpha + \gamma)}$$

Similarly from $\triangle$QAB, we get

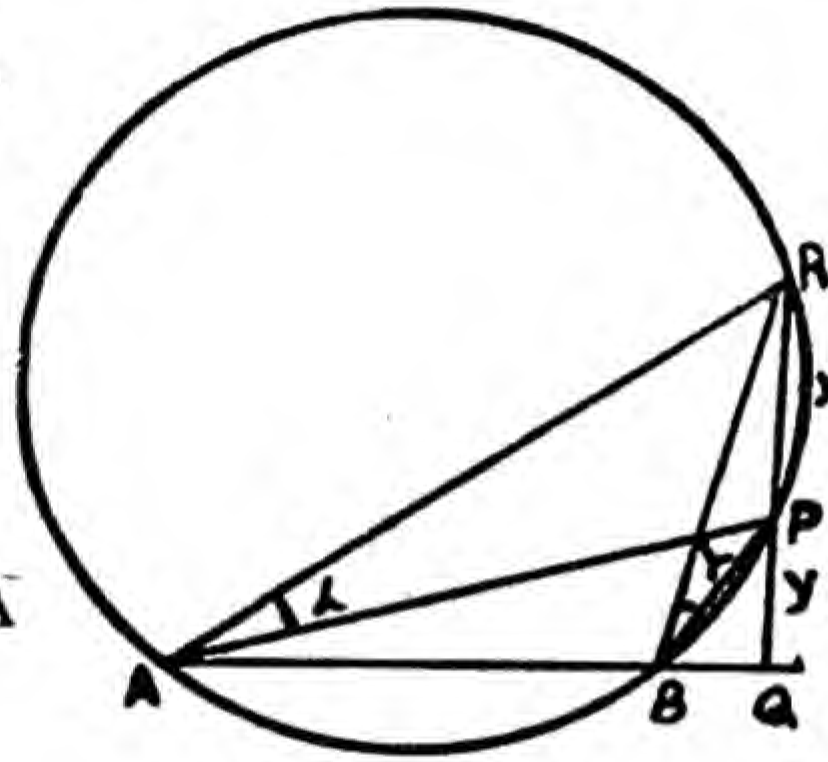
$$\frac{AQ}{AB} = \frac{\sin \delta}{\sin (180 - \beta - \delta)} \quad i.e., \frac{\sin \delta}{\text{sia } (\beta + \delta)}$$

$$\therefore AQ = \frac{a \sin \delta}{\sin (\beta + \delta)}$$

Thus in the $\triangle$PAQ sides AP, AQ and the included angle are known. Applying Napier's analogy, PQ will be determined.

Note. If P, Q, B, A are in the same plane, then $\angle$PAQ $= \alpha - \beta$. But in the above example they are not in the same plane; hence $\angle$PAQ has to be measured separately.

Ex. 7. A flagstaff PR on a tower PQ subtends the same angle α at two places A and B, distance a apart in the horizontal plane, in a line with the foot of the tower. The tower subtends $\angle\beta$ at A. Find the height of the flagstaff and tower.



Let PR$=x$ and PQ $= y$

Since PR subtends $\angle\alpha$ at A and B, P R A B is a cyclic quadrilateral.

$$\angle ARB=\angle APB$$
$$=\angle APQ-\angle BPQ$$
$$=(90°-\beta)-\angle RAB$$
$$=(90°-\beta)-(\alpha+\beta)$$
$$=90°-2\beta-\alpha.$$

In $\triangle APR, \dfrac{x}{\sin\alpha}=\dfrac{AR}{\sin RPA}$

$$=\frac{AR}{\sin RBA}$$

and in $\triangle ARB, \dfrac{AR}{\sin RBA}=\dfrac{a}{\sin ARB}$ *i. e.* $\dfrac{a}{\sin(90°-2\beta-\alpha)}$

$\therefore x=\dfrac{a\sin\alpha}{\sin(90-2\beta-\alpha}$ *i e.* $\dfrac{a\sin\alpha}{\cos(2\beta+\alpha)}$

again, $y=$PB cos BPQ $=$ PB cos $(a+\beta)$

and $\dfrac{PB}{\sin\beta}=\dfrac{a}{\sin(90-2\beta-\alpha)}$ *i.e.* $\dfrac{a}{\cos(2\beta+\alpha)}$

$\therefore y=\dfrac{a\sin\beta\cos(\alpha+\beta)}{\cos(2\beta+\alpha)}$

EXERCISE 32.

1. The angles of elevation of a building as seen from points B and C are respectively 55° and 25°, the points B and C being at a distance of 100′ from one another in a horizontal straight line which if produced, could pass through the base of the building. Find the height of the building. (P.U.)

2. The angular elevation of the top of a tower as seen

from the top and the bottom of a building 60′ high are 50° and 75° respectively. Find the height of the tower to the nearest foot. (M. U.)

3. AB is a straight road leading to C, the foot of a tower. A being at a distance of 400 ft. from C and B, 250 ft. nearer. If the angle of elevation of the tower at B be double of the angle of elevation at A, find the height of the tower and the angle of elevation at A. (P. U. 1935)

4. The angle of elevation of a tower from a point A due south of it is x and from a point B due east of A is y. If $AB=l$, show that the height h of the tower is given by $h^2(\cot^2 y-\cot^2 x)=l^2$. (P. U. 1943)

5. At the foot of a mountain the elevation of its summit is 45°. After ascending one mile towards the mountain up an incline of 30°, the elevation is 60°. How high is the mountain? (J. & K. U. 1950)

6. A person walking along the straight bank of a river observes that an object on the other bank makes an angle of 22° 48′ with the bank. He walks a distance of 400 ft. further and observes that the object now makes an angle 69° 15′. Find the breadth of the river. (P. U. 1936 S)

7. A balloon is observed simultaneouly from three places A, B, C lying due west of it on a horizontal straight line passing directly underneath it; AB = 220 ft., BC = 100 ft. and the elevation at B and C are respectively twice and thrice that at A. Calculate the height of the balloon.

8. A vertical tower stands on a horizontal plane and is surmounted by a vertical flagstaff of height h. At a point on the plane the angle of elevation of the bottom of the flagstaff is α and that of the top of the flagstaff is β. Prove that the height of the tower is $\dfrac{h\tan\alpha}{\tan\beta-\tan\alpha}$. (P. U. 1949 S)

Evaluate this expression when $h=22$ ft. $\alpha=30°\ 5'$ and $\beta=40°$.

9. The elevation of a tower from a poiut A due east of it is observed to be 45° and from a point B due north of A to

be 30°. If AB=100 ft., find the height of the tower. (D. U. 1952)

10. A tower subtends an angle α at a point on the same level as the foot of the tower, and at a second point h ft. above the first, the depression of the base of the tower is β. Find the height of the tower. (P. U. 1937)

11. The angles of elevation of two aeroplanes one passing vertically over the other are seen by an observer to be 39° and 47° respectively. If the height of the lower aeroplane above the ground be 4049 ft. find the height of the upper aeroplane. (P. U. 1936)

12. ABCD is a rectangular floor of a hall. A pillar at C subtends 18° at A and 30° at B. Find the height of the pillar and the length of the room, given AB=48 ft. (M. U.)

13. The angular elevation of a cliff from a fixed point A is θ, and after going up to a distance of k ft. toward the top of the cliff at an angle ϕ, it is found that the angular elevation is α, show that the height of the cliff is

$$\frac{k \sin \theta \sin (\alpha-\phi)}{\sin (\alpha-\theta}.$$

(D. U. 1947)

14. The angles of elevation of the top of a tower from two points distance a and b from the base and in the same straight line with it are complementary. Prove that the height of the tower is $\sqrt{ab}$ and if θ be the angle subtended at the top of the tower by the line joining the points, then

$$\sin \theta=\frac{a-b}{a+b}.$$

(P. U. 1948)

15. A tower is observed from two stations A and B. It is found to be north of A and north west of B. B is due east of A and distant 100 ft. from it. The elevation of the tower as seen from A is the complement of the elevation as seen from B. Find the height of the tower. (P. U. 1944)

16. From a point 100 ft. above the surface of a lake, the angular elevation of the peak is found to be 15° and the angle of depression of the image of the peak is 30°. Find the height of the peak. (P. U. 1940)

17. The elevation of a tower due north of a station at A is α, and at a station B due west of A is β. Prove that its altitude is $\dfrac{AB \sin \alpha \sin \beta}{\sqrt{\sin^2 \alpha - \sin^2 \beta}}$. (D. U. 1956)

18. An observer on the top of a light house 100 ft. high finds the angles of depression of two buoys A and B on the sea to be 45° and 30° respectively and the angles subtended at the eye to be 62°. Calculate the distance between the buoys. (M. U.)

19. A man in a balloon observes that the angles of depression of an object due North is 30°. The balloon drifts 3 miles due west and the angle of depression of the object is then found to be 21°. Find the height of the balloon in miles correct to two decimal places. (M. U.)

20. A tower 51 ft. high has a mark at a height of 25 ft. from the ground, find at what distance the two parts subtend equal angles at an eye at the height of 5 ft. from the ground. (D. U 1934)

21. If in the plane quadrilateral ABCD, AB=193 ft., $\angle BAC = 37°$, $\angle CAD = 21°$, $\angle ABD = 59°$ and $\angle CBD = 23°$, find CD.

22. The stations A, B, C are in a horizontal line passing through the foot of a tower, and the angle of elevation of the top of the tower at three points to be θ, $90° - \theta$, 2θ respectively.

If $AB = a$, $BC = b$, prove that $a = 2(a+b) \cos 2\theta$ and that the height of the tower is $\frac{1}{2}\sqrt{(3a+2b)(a+2b)}$ $(30° < \theta < 45°)$. (Bom. U.)

23. A pole 100 ft. high stands in the centre of an equilateral triangle which is horizontal. From the top of the pole each side subtends an angle of 60°. Prove that the length of the side of the triangle is $50\sqrt{6}$ ft. (P. U. 1944, 56)

24. From a point 100 ft. above the surface of a lake the angular elevation of the peak is found to the 15° and the angle of depression of the image of the peak is 30°. Find the height of the peak. (P. U. 1955)

25. A statue on the top of a pillar subtends the same angle α at distances of 9 and 11 yds. from the pillar. If $\alpha = \frac{1}{10}$, find the height of the statue. (P. U.)

CHAPTER XIV

RADII OF CIRCLES CONNECTED WITH A REGULAR POLYGON

Area of a regular Polygon and a circle.

Def – Regular Polygon :– By a regular polygon is meant a polygon which has all its sides equal and all its angles equal.

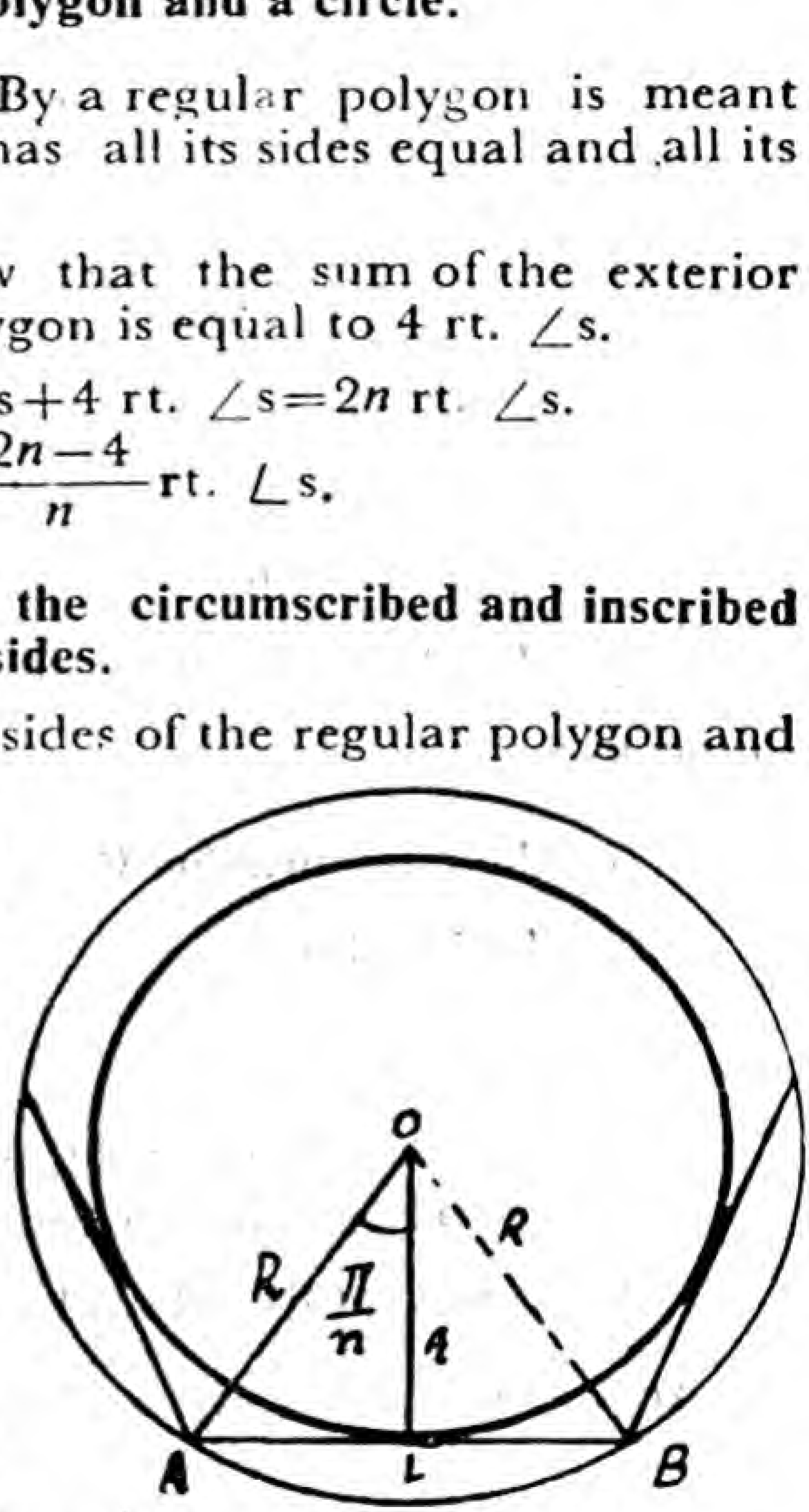
From geometry we know that the sum of the exterior angles of an n sided regular polygon is equal to 4 rt. $\angle$s.

$\therefore$ Sum of n interior angles $+4$ rt. $\angle s = 2n$ rt. $\angle$s.

$\therefore$ one interior angle $= \dfrac{2n-4}{n}$ rt. $\angle$s.

1. To find the radii of the circumscribed and inscribed circles of a regular polygon of n sides.

Let AB be one of the sides of the regular polygon and let $AB = a$. Draw the bisectors of angles A and B and let them meet at O, then O is the centre of both the incircle and the circumcircle of the polygon. Let r and R be their radii.



Draw $OL \perp AB$.

Then $OA = OB = R$ and $OL = r$.

Now, since AB is a chord of the circumcircle and $OL \perp AB$.

$\therefore$ $AL = LB = \dfrac{a}{2}$ and $\angle AOL = \angle BOL = \frac{1}{2}\angle AOB$

But the polygon is regular.

$\therefore\ \angle AOB = \dfrac{1}{n}$ of the whole angle at O.

$\therefore\ \angle AOB = \dfrac{2\pi}{n}$, or $\angle AOL = \dfrac{\pi}{n}$.

Now from $\triangle$ AOL, $\dfrac{AL}{OA} = \sin\dfrac{\pi}{n}$ *i. e.* $\dfrac{a/2}{R} = \sin\dfrac{\pi}{n}$

$$\therefore\ R = \frac{a}{2\sin\frac{\pi}{n}} = \frac{a}{2}\operatorname{cosec}\frac{\pi}{n} \qquad \ldots(1)$$

Again $\dfrac{AL}{OL} = \tan\dfrac{\pi}{n}$ *i. e.* $\dfrac{a/2}{r} = \tan\dfrac{\pi}{n}$

$$\therefore\ r = \frac{a}{2\tan\frac{\pi}{n}} = \frac{a}{2}\cot\frac{\pi}{n}. \qquad \ldots(2)$$

2. To find the area of a regular polygon of n sides in terms of

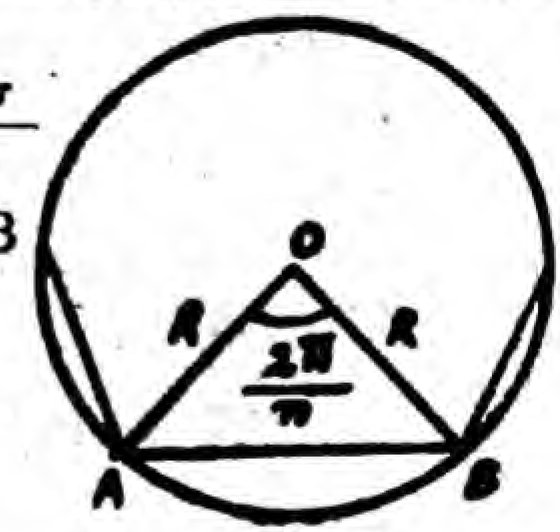
(*i*) the circumradius R. (P. U. 1955)

(*ii*) the inradius r.

(*iii*) the side a. (P. U. 1953)

(*i*) Let AB be a side of the regular polygon of n sides and O the circum-centre.

Then $OA = OB = R$ and $\angle AOB = \dfrac{2\pi}{n}$



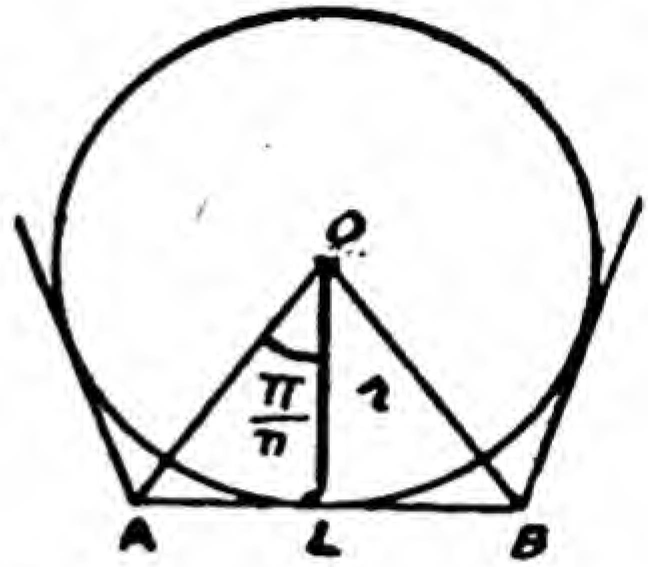
$\therefore$ Area of the polygon $= n \times$ area of $\triangle AOB$

$= n \times \frac{1}{2}\ OA.\ OB \sin \angle AOB$

$= n \times \frac{1}{2}\ R^2 \sin\dfrac{2\pi}{n} = \dfrac{nR^2}{2}\sin\dfrac{2\pi}{n}.$

(*ii*) Let AB be a side of the regular polygon and O its in-centre. Draw $OL \perp AB$.

Then $OL = r$.



Since OL bisects AB as well as $\angle AOB$, therefore $AL = \frac{1}{2}$ AB and $\angle AOL = \frac{1}{2} \angle AOB = \frac{1}{2}\left(\frac{2\pi}{n}\right) = \frac{\pi}{n}$.

$\therefore$ Area of the polygon

$= n \times$ area of $\triangle AOB$

$= n \times \frac{1}{2}$ AB. OL

$= n \times AL \times OL$

$= n.r \tan \frac{\pi}{n}.r \qquad \left(\because \frac{AL}{OL} = \tan \frac{\pi}{n}\right)$

$= nr^2 \tan \frac{\pi}{a}$.

(*iii*) Let AB $(=a)$ be a side of the regular polygon of n sides and O the incentre [See fig. for (*ii*)].

Draw $OL \perp AB$, then OL bisects AB as well as $\angle AOB$.

$\therefore \; AL = \frac{1}{2} AB = \frac{a}{2}$

and $\angle AOL = \frac{1}{2} \angle AOB = \frac{1}{2}\left(\frac{2\pi}{n}\right) = \frac{\pi}{n}$

Area of the polygon $= n \times$ area of $\triangle AOB$

$= n \times \frac{1}{2}$ AB. OL $= n$ AL. OL

$= n \times \frac{a}{2} \times \frac{a}{2} \cot \frac{\pi}{n} \qquad \left(\therefore \frac{AL}{OL} = \tan \frac{\pi}{n}\right)$

$= \frac{1}{4} na^2 \cot \frac{\pi}{n}$

Ex. 1. If an equilateral triangle and a regular hexagon have the same perimeter, prove that their areas are 2 : 3.

(P.U. 1940 S)

Let each side of the triangle $= a$, and that of the hexagon $= a_1$.

then $3a=6a_1$, $\therefore a=2a_1$

Now area of a regular polygon of n sides$=\dfrac{na^2}{4}\cot\dfrac{\pi}{n}$

(Art. 2 (*iii*))

$$\therefore \frac{\text{Area of triangle}}{\text{Area of hexagon}}=\frac{\dfrac{3}{4}a^2\cot\dfrac{\pi}{3}}{\dfrac{6}{4}a_1^2\cot\dfrac{\pi}{6}}$$

(Putting $n=3$ and 6)

$$=\frac{\dfrac{3}{4}\cdot\ 4a_1^2.\ \dfrac{1}{\sqrt{3}}}{\dfrac{6}{4}\ a_1^2\sqrt{3}}=\tfrac{2}{3}.$$

Ex. 2. The sides of a triangle are respectively a side of a regular pentagon, hexagon and decagon inscribed in a circle, prove that the triangle is right-angled.

$$R=\frac{a}{2\sin\dfrac{\pi}{n}}\ \textit{i.e.}\ a=2R\sin\frac{\pi}{n}$$

Obviously the sides of a pentagon will be the longest and

$$=2R\ \sin\frac{\pi}{5}$$

$$=2R\sin 36^\circ=2R\ \frac{\sqrt{10-2\sqrt{5}}}{4}$$

$(\because\ \sin 36^\circ=\sqrt{(1-\cos^2 36^\circ)}$

$$=R\ \frac{\sqrt{10-2\sqrt{5}}}{2}$$

Side of a regular hexagon $=2R\sin\frac{\pi}{6}=2R\sin 30°=R$

and side of a regular decagon $=2R\sin\frac{\pi}{10}=2R\sin 18°$

$$=2R.\frac{\sqrt{5}-1}{4}$$

$$=\frac{R(\sqrt{5}-1)}{2}.$$

Now, (side of hexagon)^2 + (side of decagon)^2

$$=R^2+\frac{R^2(\sqrt{5}-1)^2}{4}=\frac{4R^2+R^2(6-2\sqrt{5})}{4}$$

$$=\frac{R^2(10-2\sqrt{5})}{4}=\text{(side of pentagon)}^2$$

Hence the $\triangle$ is right-angled.

Exercise 33.

1. One side of a regular decagon is 4 inches, find the radii of the inscribed and circumscribed circles and area of the polygon. (P. U. 1937)

2. If regular octagons be described about and in a given circle, find the ratio of their areas.

3. If a be the side of a regular polygon of n sides, R and r the radii of the circumcircle and the incircle of the polygon, prove that $R+r=\frac{a}{2}\cot\frac{\pi}{2n}$.

4. If R, r be the radii of the circumscribed and inscribed circles of a regular polygon and R', r' those of the regular polygon of the same area but double the number of sides, show that $R'=R\sqrt{Rr}$ and $r'=\sqrt{\frac{r}{2}(R+r)}$. (D. U.)

5. Prove that the perimeters of the circumscribing polygon, the circle and the inscribed polygon are in the ratio

$\sec\frac{\pi}{n} : \frac{\pi}{n}\operatorname{cosec}\frac{\pi}{n} : 1$ and that the areas of the polygons are in the ratio $1 : \cos^2\frac{\pi}{n}$.

6. The area of a regular polygon of n sides inscribed in a circle is to that of the same number of sides circumscribing the same circle as 3 : 4, find the value of n. (P. U. 1941)

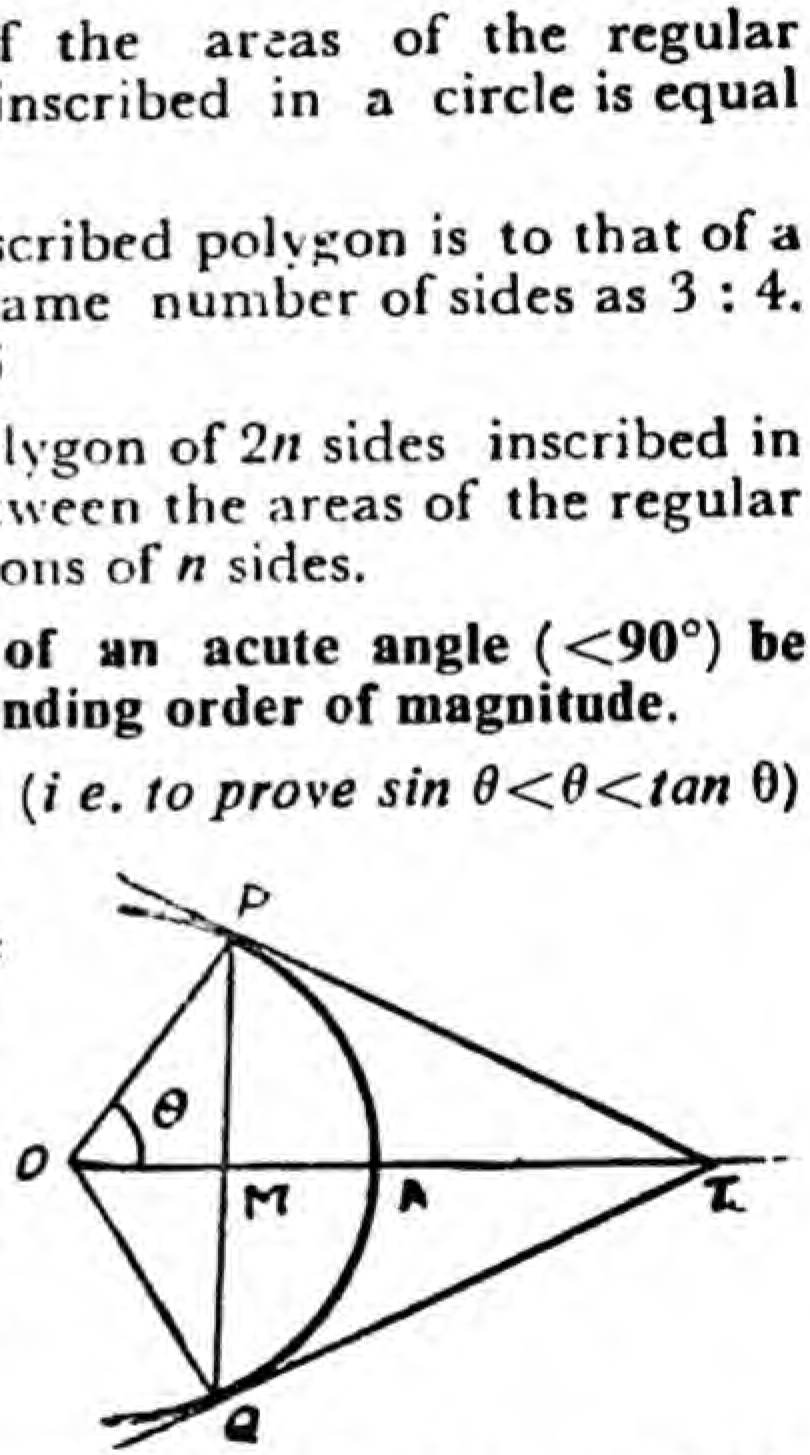
7. Show that the ratio of the areas of the regular octagons circumscribed to, and inscribed in a circle is equal to $2\sqrt{2} : (\sqrt{2}-1)$.

8. The area of a regular inscribed polygon is to that of a circumscribing polygon of the same number of sides as 3 : 4. Show that the number of sides is 6

9. The area of a regular polygon of $2n$ sides inscribed in a circle is a mean proportional between the areas of the regular inscribed and circumscribed polygons of n sides.

3. If the circular measure of an acute angle ($<90°$) be θ then $\sin\theta$, θ and $\tan\theta$ are in ascending order of magnitude.

(i e. to prove $\sin\theta<\theta<\tan\theta$)



Let AOP be any acute angle and θ the number of radians in it.

With O as centre and any radius draw a circle cutting OP, OA in P and A. Draw PM$\perp$OA and produce it to cut circle again in Q.

Draw the tangent at P and produce it to meet OA in T. Join TQ and OQ.

The right angled $\triangle$s OPM and OQM are congruent.

$\therefore$ MP=MQ and arc PA=arc QA.

Again, from the congruent $\triangle$s OPT and OQT, TP=TQ.

Now *assuming* that arc PAQ<PT+TQ, we have

Chord PQ<arc PQ<PT+TQ

$\therefore$ 2MP<2 arc PA<2PT

$\therefore$ MP<arc PA<PT

$$\therefore \frac{MP}{OP} < \frac{\text{arc PA}}{OP} < \frac{PT}{OP} \qquad \text{(Dividing by OP)}$$

i e. $\sin\theta < \theta < \tan\theta$.

4. If θ be measured in radians, to prove that

$$\mathop{\text{Lt}}_{\theta \to 0} \frac{\sin\theta}{\theta} = 1.$$

$\because \sin\theta < \theta < \tan\theta$, (when $\theta < \frac{\pi}{2}$)

Dividing by $\sin\theta$, we get

$$1 < \frac{\theta}{\sin\theta} < \frac{1}{\cos\theta}$$

or taking reciprocals, $1 > \frac{\sin\theta}{\theta} > \cos\theta$.

i. e., $\frac{\sin\theta}{\theta}$ lies between 1 and $\cos\theta$.

But when $\theta \to 0$, $\cos\theta \to 1$ and therefore $\frac{\sin\theta}{\theta}$ which lies between 1 and $\cos\theta$ also approaches unity.

$$\therefore \mathop{\text{Lt}}_{\theta \to 0} \frac{\sin\theta}{\theta} = 1.$$

Cor 1. If θ be measured in radians, $\mathop{\text{Lt}}_{\theta \to 0} \frac{\tan\theta}{\theta} = 1$.

$$\mathop{\text{Lt}}_{\theta \to 0} \frac{\tan\theta}{\theta} = \mathop{\text{Lt}}_{\theta \to 0} \frac{\sin\theta}{\cos\theta} \cdot \frac{1}{\theta} = \mathop{\text{Lt}}_{\theta \to 0} \frac{\sin\theta}{\theta} \cdot \frac{1}{\cos\theta} = 1.$$

Cor. 2. If θ be measured in radians then

$$(i)\ \underset{n\to\infty}{\text{Lt}} \frac{\sin\frac{\theta}{n}}{\frac{\theta}{n}} = 1 \text{ and } (ii)\ \underset{n\to\infty}{\text{Lt}} \frac{\tan\frac{\theta}{n}}{\frac{\theta}{n}} = 1$$

$$\because \frac{\theta}{n} \to 0 \text{ as } n \to \infty$$

Cor. 3. If θ is the number of radians in an angle which is very small, then $\sin\theta = \theta$ and $\tan\theta = \theta$.

Ex. 1. Show that $\underset{n\to\infty}{\text{Lt}}\, n \sin\frac{\theta}{n} = \theta$, when θ is given in radians.

$$\underset{n\to\infty}{\text{Lt}}\, n\sin\frac{\theta}{n} = \underset{n\to\infty}{\text{Lt}} \frac{\sin\frac{\theta}{n}}{\frac{\theta}{n}}.\,\theta = 1.\theta = \theta.$$

Ex. 2 Find $\underset{x\to 0}{\text{Lt}} \frac{\sin x^\circ}{x}$

Here x° must be changed into radians.

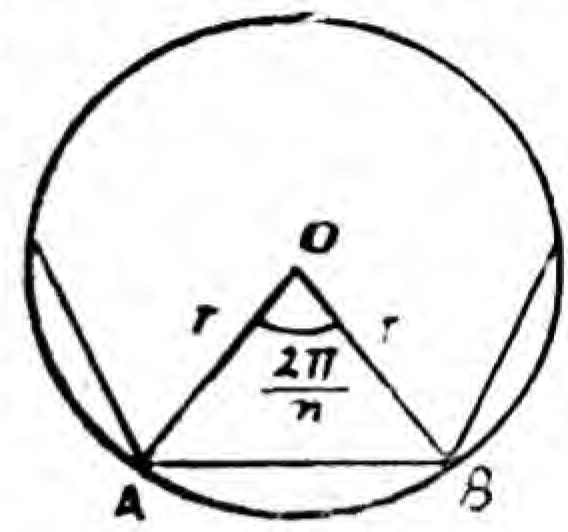
$\because 180^\circ = \pi$ radians

$\therefore x^\circ = \frac{\pi}{180} x$ radians

$$\therefore \underset{x\to 0}{\text{Lt}} \frac{\sin x^\circ}{x} = \underset{x\to 0}{\text{Lt}} \frac{\sin\frac{\pi x}{80}}{x} = \underset{x\to 0}{\text{Lt}} \frac{\sin\frac{x\pi}{180}}{\frac{\pi x}{180}} \times \frac{\pi}{180} = \frac{\pi}{180}.$$

5. (*a*) **To prove that the area of a circle of radius R is πR^2.** (P. U. 1955)

Let O be the centre and R the radius of the circle circumscribing a regular polygon of n sides. Let AB be a side of the polygon. Join OA and OB. Then $\angle AOB = \dfrac{\pi}{n}$



$\therefore$ Area of the polygon $= n \times \triangle AOB$

$$= n \times \tfrac{1}{2}\, OA.\, OB \sin \angle AOB$$

$$= \frac{n}{2} R^2 \sin \frac{2\pi}{n}.$$

Now let the number of sides of this polygon be increased indefinitely, the polygon always remaining regular. Then the polygon tends to coincide with the circle and the difference between the area of the polygon and the circle gets smaller and smaller and approaches zero.

$$\therefore \text{ Area of the circle} = \underset{n\to\infty}{\text{Lt.}}\ \tfrac{1}{2} n\, R^2 \sin \frac{2\pi}{n}$$

$$= \underset{n\to\infty}{\text{Lt}}\ \frac{n}{2} R^2 \frac{\sin \frac{2\pi}{n}}{\frac{2\pi}{n}} \times \frac{2\pi}{n}$$

$$= \underset{n\to\infty}{\text{Lt}}\ \pi R^2 \frac{\sin \frac{2\pi}{n}}{\frac{2\pi}{n}} = \pi R^2 .\, 1 = \pi R^2,$$

(b) **To prove that the circumference of a circle of radius R is 2π R.**

We know that the perimeter of a regular polygon of n sides in terms of the radius R of the circumcribed circle

$$= n.\, 2R \sin \frac{\pi}{n}$$

Now let the number of sides of the polygon be increased indefinitely. the polygon always remaining regular. The

perimeter of the polygon tends to coincide with the circumference of the circle.

Hence the circumference of the circle

$$=\underset{n\to\infty}{\text{Lt}}\ n\ 2R\ \sin \pi/n$$

$$=\underset{n\to\infty}{\text{Lt.}}\ n.\ 2R\ \frac{\sin \pi/n}{\pi/n}.\ \frac{\pi}{n}$$

$$=\underset{n\to\infty}{\text{Lt.}}\ 2\pi R.\ \frac{\sin \pi/n}{\pi/n}=2\pi\ R.$$

Second Method. We can also deduce the area of the circle from the area of the circumscribed polygon.

(*a*) The area of the circumscribed polygon in terms of the radius of the inscribed circle $=nr^2 \tan\frac{\pi}{n}$ [Art. 2 (*iii*)]

Now let the number of sides of the polygon be increased indefinitely, the polygon always remaining regular. Thus the polygon tends to coincide with the circle and the difference between the area of the polygon and the circle gets smaller and smaller and aproaches zero.

$$\therefore \text{ Area of the circle}=\underset{n\to\infty}{\text{Lt.}}\ nr^2 \tan\frac{\pi}{n}$$

$$=\underset{n\to\infty}{\text{Lt}}\ \pi r^2.\ \frac{\tan\frac{\pi}{n}}{\frac{\pi}{n}}=\pi r^2$$

(*b*) We know that the perimeter of a regular polygon of n sides in terms of the radius r of the inscribed circle

$=n.\ 2r\ \tan\frac{\pi}{n}$. Now let the number of sides of the polygon be increased indefinitely, the polygon always remaining regular. The perimeter of the polygon tends to coincide with the circumference of the circle,

$\therefore$ Circumference of the circle

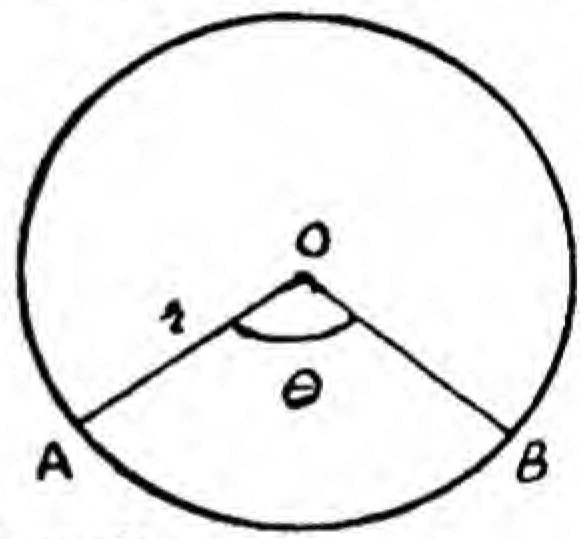
$$=\operatorname*{Lt}_{n\to\infty} n.\ 2r\tan\frac{\pi}{n}$$

$$=\operatorname*{Lt}_{n\to\infty} 2r.\ \pi.\ \frac{\tan\frac{\pi}{n}}{\frac{\pi}{n}}$$

$$=2\pi r.\ 1=2\pi r.$$

6. To find the area of a sector of a circle.

Let AB be an arc of a circle, centre O and radius r, and let $\angle AOB=\theta$ radians



$\therefore \dfrac{\text{area of the sector AOB}}{\text{area of the circle}}=\dfrac{\angle AOB}{2\pi}$ i e $\dfrac{\theta}{2\pi}$

Hence the area of the sector AOB

$$=\frac{\theta}{2\pi}\times\pi r^2$$
$$=\tfrac{1}{2}r^2\theta.$$

Cor. 1. The area of a sector $=\frac{1}{2}r^2\theta=\frac{1}{2}r\times r\theta.$

$$=\tfrac{1}{2}\,r\times\text{arc AB}\left(\because \frac{l}{r}=\theta^{\circ}\right)$$
$$=\tfrac{1}{2}\text{ arc AB}\times\text{radius.}$$

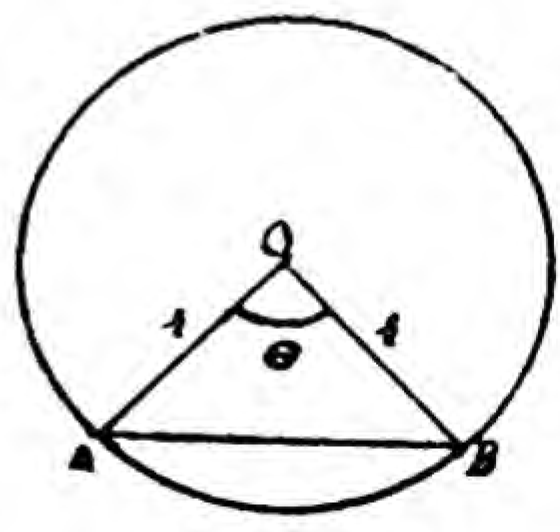
Cor. 2. To find the area of a segment of a circle.

Area of segment APBA=

area of sector$AOB-\triangle AOB$

$$=\tfrac{1}{2}r^2\theta-\tfrac{1}{2}\,r.\,r.\sin\theta$$
$$=\tfrac{1}{2}r^2\,(\theta-\sin\theta),$$

where θ is given in radians.



EXERCISE 34.

1. A chord 18 inches long is placed in a circle of radius 25 inches, find (*i*) the angle subtended at the centre. (*ii*) the length of the arc, (*iii*) the area of the sector and the segment respectively. (P. U. 1935)

2. Four equal circles of radius a touch each each other, show that the area enclosed between them is $a^2(4-\pi)$.

3. Three equal circles of radius a touch each other, show that the area between them is $\left(\sqrt{3}-\frac{\pi}{2}\right)a^2$

4. Prove that when θ is small,

$$(i)\quad \sin\left(\frac{\pi}{4}+\theta\right)=\frac{1+\theta}{\sqrt{2}} \text{ approximately.}$$

$$\text{and}\quad (ii)\quad \cos\theta=1-\frac{\theta^2}{4}.$$

5. Show that (*i*) $\underset{x\to\theta}{\text{Lt}}\dfrac{\sin x'}{x'}=\dfrac{\pi}{10800}$

$$(ii)\quad \underset{\theta\to 0}{\text{Lt.}}\frac{\sin b\theta}{\sin a\theta}=\frac{b}{a}\quad (iii)\quad \underset{n\to\infty}{\text{Lt}}\; n\sin\frac{\pi}{n}=\pi$$

6. Find approximately (*i*) the value of sin 10" to 6 places of decimals, and (*ii*) the value of sin 1° to 5 places of decimals.

7. The perimeter of a sector of circle is 10 ft., if the radius of the circle is 3 ft. find the area of the sector.

8. **Euler's theorem** Prove that

$$\sin\theta=\theta\cos\frac{\theta}{2}\cos\frac{\theta}{2^2}\cos\frac{\theta}{2^3}\;\ldots\ldots ad \text{ infinitum}$$

(P. U. 1940)

[Hint. $\sin\theta=2\sin\frac{\theta}{2}\cos\frac{\theta}{2}=2.2\sin\frac{\theta}{2^2}\cos\frac{\theta}{2^2}\cos\frac{\theta}{2}$

$$=2^2 \sin\frac{\theta}{2^2} \cos\frac{\theta}{2} \cos\frac{\theta}{2^2}$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$=2^n \sin\frac{\theta}{2^n} \cdot \cos\frac{\theta}{2} \cos\frac{\theta}{2^2} \cdots\cdots \cos\frac{\theta}{2^n}$$

When $n \to \infty$, Lt. $2^n \sin\frac{\theta}{2^n}=\theta$

$$\therefore \sin\theta=\theta \cos\frac{\theta}{2} \cos\frac{\theta}{2^2} \cdots\cdots \text{to } \infty]$$

(9) The perimeter of a certain sector of a circle is 20 ft; if the radius of the circle is 6 ft., find the area of the sector.

(D. H. S. 1958)

J. & K. University Papers

K. U. 1955

1. (*a*) Show that the circular measure of an angle equals the ratio which the length of the arc of a circle, subtending that angle at the centre bears to the radius of the circle.

(*b*) Find the angle in radians subtended at the centre of a circle of radius 5 ft. by an arc 11 inches long. Convert it into degrees and minutes.

(*c*) If A is in the fourth quadrant and

$\text{Cos A} = \frac{5}{13}$, find the value of $\frac{13 \sin A + 5 \text{ Sec A}}{5 \tan A + 6 \text{ Cosec A}}$.

2 (*a*) Prove that for all values of θ,
$\sin(\pi+\theta) = -\sin\theta$. Draw four figures.

(*b*) Draw the graph of Sin x, as x varies from 0 to 2π. Verify the result obtained in (*a*) above.

3. Prove any three of the following :—

(*a*) $$\frac{1}{\text{Sec } x - \tan x} - \frac{1}{\text{Cos } x} = \frac{1}{\text{Cos } x} - \frac{1}{\text{Sec } x + \tan x}$$

(*b*) $\text{Sin } 17^\circ 26' \text{ Cos } 12^\circ 34' + \text{Sin } 72^\circ 24' \text{ Sin } 12^\circ 34' = \frac{1}{2}$

(*c*) $$\text{Sin } 3\text{A} + \text{Sin } 2\text{A} - \text{Sin A} = 4 \text{ Sin A Cos } \frac{\text{A}}{2} \text{ Cos } \frac{3\text{A}}{2}$$

(*d*) $\text{Sin } 18^\circ = \frac{\sqrt{5}-1}{4}$

4. (*a*) Obtain the general expression for all angles having a given tangent. Hence find the period of tangent.

(*b*) Solve one of the following equations giving the solutions in a general form. Check your answers.

(*i*) $\tan^2\theta+\cot^2\theta=2$

(*ii*) $\sin\theta+\sqrt{3}\cos\theta=1$

5. (*a*) Prove that, in a triangle ABC,

$$\sin\frac{A}{2}=\sqrt{\frac{(s-b)(s-c)}{bc}}$$

(*b*) Show that (*i*) $r_1=\dfrac{\Delta}{s-a}$

and (*ii*) $\dfrac{1}{r_1}+\dfrac{1}{r_2}+\dfrac{1}{r_3}=\dfrac{1}{r}$.

with the usual notation.

6. (*a*) A man observes that the elevation of a mountain top is 30° and after walking a mile directly towards it on a level ground the elevation is 75°. Find the height of the mountain in feet, correct to four significant figures.

(*b*) Find the greatest angle in the triangle whose sides are 40′, 21′ and 23′ correct to the nearest second.

K. U. 1956

1. (*a*) Define a radian and show that it is a constant angle.

(*b*) If $\cos A=2\sin A$, find $\operatorname{cosec} A$, A being in the third quadrant.

(*c*) Prove that $\sec A-\tan A=\dfrac{\cos A}{1+\sin A}$

2. (*a*) By drawing figures in several quadrants, prove that $\tan\left(\dfrac{\pi}{2}+\theta\right)=-\cot\theta$

(*b*) Draw the graph of tan θ, as θ varies from θ to 2π.

(*c*) Can you illustrate the formula in (*a*) from the graph? If so, how?

3. (*a*) Prove geometrically that

$$\cos(A+B)=\cos A\cos B-\sin A\sin B,$$

$(A+B)$ being in the second quadrant.

(*b*) If $A+B+C=\pi$, prove that

(*ii*) $\sin 2A+\sin 2B+\sin 2C=4\sin A\sin B\sin C$

Or

(*iii*) Prove that $2\tan 50^\circ+\tan 20^\circ=\tan 70^\circ$

4. (*a*) Find the general expressin for all angles having the same cosine.

(*b*) Solve one of the following equations giving the general value of θ :—

(*i*) $\sin\theta-\cos\theta=\dfrac{1}{\sqrt{2}}$

(*ii*) $\operatorname{cosec}^2\theta=4$

5. In any triangle, prove three of the following relations :—

(*i*) $\dfrac{a}{\sin A}=\dfrac{b}{\sin B}=\dfrac{c}{\sin C}$

(*ii*) $r\,r_1\,r_2\,r_3=\triangle^2$,

(*iii*) $r_1=s\tan\dfrac{A}{2}$

(*iv*) $r_1+r_2+r_3-r=4R$.

6. (*a*) A person standing on the bank of a river finds that the angle of elevation of the top of a cliff on the opposite bank is 60°; on going back 100 yds. he finds that the angle of elevation is only 30°. Find the height of the top of the cliff and breadth of the river.

(*b*) Solve the triangle, given that :

$b=41$, $c=36.4$ and angle $B=42^\circ\,27'$,

Can it admit of two solutions?

K. U. 1957

1. (*a*) Define a radian, show that it is a constant angle and express it in sexagesimal measure correct to the nearest second.

What is the difference between π and π radians ?

(*b*) If G, D, C be the number of grades, degrees and radians in any angle, prove that

$$\frac{D}{9}=\frac{G}{10}=\frac{20C}{\pi}$$

2. (*a*) Prove that $\sec^2\theta=1+\tan^2\theta$ where θ is any angle.

(*b*) Prove the identity $(\sin x+\sec x)^2+(\operatorname{cosec} x+\cos x)^2=(1+\sec x \operatorname{cosec} x)^2$.

(*c*) Two posts of the same height stand on either side of a road 120 ft. wide; at a point in the road between the posts, the elevations of the tops of the pillars are 60° and 30°. Find height of the posts and the position of the point.

3. (*a*) Prove that for all values of θ, $\tan(\pi+\theta)=\tan\theta$.

(*b*) Draw the graph of $\tan\theta$ for $0\leqslant\theta\leqslant2\pi$ and find from the graph the values of θ which satisfy the equation $\tan\theta=\cot\theta$.

(*c*) Prove that $\tan\theta\tan\left(\frac{\pi}{2}\pm\theta\right)\pm1=0$

4. (*a*) If $\alpha+\beta+\gamma=\frac{\pi}{2}$, prove that

$$\tan\alpha\tan\beta+\tan\beta\tan\gamma+\tan\gamma\tan\alpha=1.$$

(*b*) Find the circular functions of 18°.

(*c*) Prove that $\cos 20^\circ\cos 40^\circ\cos 60^\circ\cos 80^\circ=\frac{1}{16}$.

5. (*a*) To prove that in any $\triangle ABC$, $\dfrac{\tan\frac{B-C}{2}}{\tan\frac{B+C}{2}}=\dfrac{b-c}{b+c}$

(*b*) If a, b, c are in H. P., prove that $\sin^2 \frac{A}{2}$,

$\sin^2 \frac{B}{2}$, $\sin^2 \frac{C}{2}$ are also in H. P.

(*c*) Solve the equation $\sin 4\theta = \sin \theta$.

6. (*a*) If $a = 182{\cdot}5$, $b = 82{\cdot}5$, $A = 72° \, 15'$, solve the triangle.

(*b*) Prove the formula $R = \frac{a}{2 \sin A} = \frac{b}{2 \sin B} = \frac{c}{2 \sin C}$, where R is the circumradius of a triangle ABC.

K. U. 1958

1. (*a*) Show that the length of an arc subtending an angle θ radians at the centre of a circle of radius r, is $r\theta$.

(*b*) A pendulum 8 ft. long oscillates through an angle of 9°; what is the length of the path its extremity describes between the extreme positions ?

(*c*) The angles of a quadrilateral are in A. P. and the greatest is double the least; express the least angle in degrees and grades.

2. (*a*) Construct angles between 0° and 360° whose tangent is $\frac{2}{3}$ and find their Secants and Cosecants.

(*b*) Prove that $(\tan \theta + \sec \theta)^2 = \frac{\text{Cosec } \theta + 1}{\text{Cosec } \theta - 1}$

(*c*) In a cyclic quadrilateral ABCD, show that :

$\text{Cos } A + \text{Cos } C = 0$ and $\text{Cos } B + \text{Cos } D = 0$.

3. (*a*) Two men A and B, 1360 yds. apart observe an aeroplane C at the same instant and find the respective angles of elevations to be 45° and 60°. If the plane ABC is vertical, find the height of the aeroplane.

(*b*) Draw the graphs of $\tan \theta$ and $\cot \theta$ between $\theta = 0$ and $\theta = \pi$ and from your graph find the values of θ which satisfy $\tan \theta = \cot \theta$.

4. (*a*) Prove that $\text{Cos }(A+B) \text{ Cos }(A-B) = \text{Cos}^2 A - \text{Sin}^2 B$

(*b*) Prove that $\text{Sin } 70° - \text{Cos } 80° = \text{Cos } 40°$.

(c) Prove that, if $A+B+C=180°$, then

$$\frac{\text{Sin } 2A+\text{Sin } 2B+\text{Sin } 2C}{\text{Sin } A+\text{Sin } B+\text{Sin } C}=8 \text{ Sin } \frac{A}{2} \text{ Sin } \frac{B}{2} \text{ Sin } \frac{C}{2}$$

5. (a) Solve $\text{Sin } \theta+\text{Sin } 2\theta+\text{Sin } 3\theta=0$.

(b) In any triangle ABC, prove that $\text{Cos} \frac{A}{2}=\sqrt{\frac{s(s-a)}{bc}}$ where $a+b+c=2s$.

(c) Prove that $\text{Sin } A+\text{Sin } B+\text{Sin } C=\frac{s}{R}$ in any $\triangle$ ABC where R is the Circum-radius and $a+b+c=2s$.

6. (a) Given $\log 2=\cdot30103$, find the number of digits in 2^{23}.

(b) If $A=50°$, $b=1071$, $a=873$; find to the nearest second, angle B. Given $\log 1\cdot071=\cdot029789$, L sin $50°=9\cdot884254$, L Sin $70°=9\cdot972986$, L Sin $70°\ 1'=9\cdot973032$, $\log 8\cdot73=\cdot941014$.

K. U. 1959

1. (a) Prove that the radian is a constant angle

(b) Show that $\frac{\tan A+\text{Sec } A-1}{\tan A-\text{Sec } A+1}=\frac{1+\text{Sin } A}{\text{Cos } A}$

2. (a) Trace the changes in the sign and magnitude of the trigonometrical ratios of an angle, as the angle increases from $0°$ to $360°$.

(b) Find a solution of the equation $3\tan\theta+\cot\theta=5\text{cosec } \theta$.

3. (a) Prove geometrically that $\cos (A-B)=\text{Cos } A \text{ Cos } B+\text{Sin } A \text{ Sin } B$.

(b) Find the expansion of cos 3 A.

4. (a) In a $\triangle$ABC if $a \cos^2 \frac{C}{2}+c \cos^2 \frac{A}{2}=\frac{3b}{2}$, show that the sides of the triangle are in A. P.

(b) Prove that $\log_a{}^m=\log_b{}^m\times\log_a{}^b$

5. (*a*) If $A+B+C=180°$, Prove that

$$\text{Sin}^2\frac{A}{2}+\sin^2\frac{B}{2}+\sin^2\frac{C}{2}=1-2\,\text{Sin}\,\frac{A}{2}\,\text{Sin}\,\frac{B}{2}\,\text{Sin}\,\frac{C}{2}$$

(*b*) In a $\triangle$ ABC prove that

$$R=\frac{a}{2\,\text{Sin}\,A}.$$

K. U. 1960

1. (*a*) Prove that $(1+\cot A+\tan A)(\sin A-\cos A)=\dfrac{\text{Sec}\,A}{\text{cosec}^2 A}-\dfrac{\text{cosec}\,A}{\text{Sec}^2 A}$

(*b*) From the top of a cliff, 200 feet high, the angles of depression of the top and bottom of a tower are observed to be 30° and 60° respectively; find the height of the tower.

2. (*a*) Prove geometrically that $\sin(A+B)=\sin A\cos B+\cos A\sin B$.

(*b*) Show that $\dfrac{\sin^2 A-\sin^2 B}{\sin A\cos A-\sin B\cos B}=\tan(A+B)$

3 (*a*) If $A+B+C=180°$ then show that
$$\sin^2 A+\sin^2 B+\sin^2 C=2+2\cos A\cos B\cos C.$$

(*b*) Solve the equation : $\sin\theta+\sin 7\theta=\sin 4\theta$

4. (*a*) Prove that (*i*) $\log_a\left(\dfrac{m}{n}\right)=\log_a{}^m-\log_a{}^n$,

(*ii*) $\log_a(m^n)=n\log_a{}^m$.

(*b*) Show that in any $\triangle$ ABC, $\cos C=\dfrac{a^2+b^2-c^2}{2ab}$

5. If $b=\sqrt{3}$, $c=1$ and $A=30°$, then solve the $\triangle$ABC.

(*b*) If r be the radius of the incircle of the triangle ABC, then show that $r=\dfrac{\Delta}{s}$, where Δ and s denote respectively the area and the semi-perimeter of the triangle ABC.

K. U. 1961

1. (*a*) Prove that

$$\sin A = \frac{2 \tan A/2}{1+\tan^2 A/2} \text{ and } \cos A = \frac{1-\tan^2 A/2}{1+\tan^2 A/2}.$$

(*b*) The shadow of a tower standing on a level plane is found to be 60 feet longer when the sun's altitude is 30° than when it is 45°. Prove that the height of the tower is $30 (1+\sqrt{3})$ feet.

2. (*a*) Prove that $\sin (A+B) \sin (A-B) = \sin^2 A - \sin^2 B$ and $\cos (A+B) \cos (A-B) = \cos^2 A - \sin^2 B$.

(*b*) Show that $1+\tan A \tan A/2 = \tan A \cot A/2 - 1 = \sec A$.

3. (*a*) If $A+B+C=180°$, prove that
$\tan A/2 \tan B/2 + \tan B/2 \tan C/2 + \tan C/2 \tan A/2 = 1$

(*b*) Solve the equation $\sin \theta + \sin 5\theta = \sin 3\theta$.

4. (*a*) Having given $\log 3 = \cdot 4771213$, find the number of digits in 3^{62}.

(*b*) In any triangle ABC, prove that

$$\frac{a \sin (B-C)}{b^2-c^2} = \frac{b \sin (C-A)}{c^2-a^2} = \frac{c \sin (A-B)}{a^2-b^2}.$$

5. (*a*) Show that in any triangle ABC,

$$\tan (B-C)/2 = \frac{b-c}{b-c} \cot A/2.$$

(*b*) If R and *r* denote respectively the radii of the circumcircle and the incircle of any triangle ABC, prove that

$$1/bc + 1/ca + 1/ab = 1/2Rr.$$

Delhi Higher Secondary Examination

Papers 1957

1. (*a*) Define a radian. Prove that it is a constant angle.

(*b*) A cow is tied to a post by a rope. If the cow moves along a circular path always keeping the rope tight, and describes 44 feet when it has traced out 72° at the centre, find the length of the rope.

(*c*) Find the value of sin 18°.

2. (*a*) Prove geometrically that

$$\sin(A-B)=\sin A\cos B-\cos A\sin B.$$

(*b*) If $\tan\frac{\theta}{2}=\left(\frac{1+c}{1-c}\right)^{\frac{1}{2}}\tan\frac{\phi}{2}$, prove that

$$\cos\theta=\frac{\cos\phi-c}{1-c\cos\phi}.$$

(*c*) Prove that

$$\left(\frac{\cos A+\cos B}{\sin A-\sin B}\right)^n+\left(\frac{\sin A+\sin B}{\cos A-\cos B}\right)^n=2\cot^n\frac{A-B}{2}$$

or zero according as n is even or odd.

3. (*a*) Prove that

(*i*) $\log_a(m.n)=\log_a m+\log_a n$.

(*ii*) $\log_y x\times\log_z y\times\log_x z=1$.

(*b*) A spherical balloon whose radius is r feet subtends at an observer's eye an angle α, when the angular elevation of the centre is β. Determine the height of the centre of the balloon.

4. In any triangle ABC prove that

(*i*) $\cos\frac{A}{2}=\sqrt{\frac{s(s-a)}{bc}}$.

(*ii*) $a \cos A + b \cos B + c \cos C = 4R \sin A \sin B \sin C$.

(*iii*) a^2, b^2, c^2 are in A. P. if

$$\frac{\sin A}{\sin C} = \frac{\sin (A-B)}{\sin (B-C)}.$$

[The symbols have their usual meanings.]

5. (*a*) Find the general expression for all angles having the same sine.

(*b*) Solve $\tan\left(\frac{\pi}{4}+\theta\right)+\tan\left(\frac{\pi}{4}-\theta\right)=4$.

(*c*) Prove that $\tan 70 = \tan 20° + 2 \tan 50°$.

6. (*a*) Four equal circles each of radius a touch one another. Find the area between them.

(*b*) If $\tan \phi = \frac{a-b}{a+b} \cot \frac{C}{2}$, prove that

$$c = (a+b) \frac{\sin \frac{C}{2}}{\cos \phi}.$$

If $a=3$, $b=1$, and $C = 53° \ 7' \ 48''$, find c without getting A and B, given $\log 2 = \cdot 30103$, $\log 25298 = 4\cdot4030862$, $\log 25299 = 4\cdot4031034$, $L \cos 26° \ 33' \ 54'' = 9\cdot9515452$, $L \tan 26° \ 33' \ 54'' = 9\cdot6989700$.

Papers 1958

1. (*a*) Prove that the number of radians in an angle subtended by an arc of a circle at the centre is $= \frac{\text{arc}}{\text{radius}}$.

(*b*) If the angles of a triangle be in A. P. and one of them be 80°, find all the three angles in radians.

(*c*) Prove that $\operatorname{cosec}^2 \theta + \sin^2 \theta$ can never be less than 2.

2 (*a*) Prove geometrically that

$$\sin (A+B) = \sin A \cos B + \cos A \sin B,$$

where A, B and A+B are acute angles.

(*b*) If $\cos(X+Y)\sin(Z+U)=\cos(X-Y)\sin(Z-U)$, prove that $\cot X \cot Y \cot Z=\cot U$.

(*c*) Prove that $\cos \alpha \cos(60^\circ-\alpha)\cos(60^\circ+\alpha)=\frac{1}{4}\cos 3\alpha$.

3. (*a*) Draw the graph of $\tan x$ between $x=0$ and $x=2\pi$ and locate on the graph the values of x for $\tan^2 x=3$.

(*b*) Two stations due south of a tower, which leans towards the north, are at distances a and b from its foot ; if α and β are the elevations of the top of the tower from these stations, prove that its inclination to the horizontal is

$$\cot^{-1}\left\{\frac{b\cot\alpha-a\cot\beta}{b-a}\right\}.$$

4. In any triangle ABC, prove that

(*i*) $\tan\frac{B-C}{2}=\frac{b-c}{b+c}\cot\frac{A}{2}$.

(*ii*) $\triangle=4Rr\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}$.

(*iii*) $\frac{b^2-c^2}{a^2}\sin 2A+\frac{c^2-a^2}{b^2}\sin 2B+\frac{a^2-b^2}{c^2}\sin 2C=0$.

[The symbols have their usual meanings.]

5. (*a*) Find the general expression for all angles having the same tangent.

(*b*) Solve the equation $3\tan\theta+\cot\theta=5\operatorname{cosec}\theta$.

(*c*) Prove that $7\log_a\frac{16}{15}+5\log_a\frac{25}{24}+3\log_a\frac{81}{80}=\log_a 2$.

6. (*a*) Solve the triangle ABC, when two sides and the angle opposite to one of them are given.

(*b*) The side of a triangle are a, b, and $\sqrt{a^2+ab+b^2}$ feet ; find the greatest angle.

(*c*) 20 feet is the perimeter of a certain sector of a circle of radius 6 feet. Find the area of the sector.

Paper 1959

1. (*a*) A wire 121 in. long is bent so as to lie along the arc of a circle of radius 180 in. Find in degrees the angle subtended at the centre by the arc.

(*b*) If an angle contains D° or G^g or c radians, prove that

$$G-D=\frac{20c}{\pi}.$$

(*c*) If $\cos\theta-\sin\theta=\sqrt{2}\sin\theta$, prove that

$$\cos\theta+\sin\theta=\sqrt{2}\cos\theta.$$

2. (*a*) Prove geometrically that

$$\cos(A+B)=\cos A\cos B-\sin A\sin B.$$

(*b*) Simplify: $\dfrac{\cos(90^\circ+\theta)\sec(-\theta)\tan(180^\circ-\theta)}{\sec(360^\circ-\theta)\sin(180^\circ+\theta)\cot(90^\circ-\theta)}$.

(*c*) Find the general expression for all the angles which have the same sine.

3. (*a*) Prove that

(*i*) $\cos 4\alpha=1-8\cos^2\alpha+8\cos^4\alpha$.

(*ii*) $\cos^2 A+\cos^2 B-\cos^2 C=1-2\sin A\sin B\cos C$,

if $A+B+C=180^\circ$.

(*b*) Find the value of $\sin 18^\circ$.

4 (*a*) Prove that

(*i*) $\log_a m^n=n\log_a m$

(*ii*) $\log_a b\times\log_b a=1$.

(*b*) In a $\triangle ABC$, $b=38{\cdot}8$, $c=42{\cdot}9$ and $A=38^\circ\,16'$, find a, B and C.

5. In a triangle ABC, prove that

(*a*) $\sin\dfrac{A}{2}=\sqrt{\dfrac{(s-b)(s-c)}{bc}}$.

(*b*) $\tan\dfrac{A-B}{2}=\dfrac{a-b}{a+b}\cot\dfrac{C}{2}$.

(*c*) $(b^2-c^2)\cot A+(c^2-a^2)\cot B+(a^2-b^2)\cot C=0$.

6. (*a*) Prove that

(*i*) $(r_1+r_2)\tan\dfrac{C}{2}=c.$

(*ii*) $r=(s-a)\tan\dfrac{A}{2}.$

(The symbols have their usual meanings.)

(*b*) Find the area of a regular polygon of *n* sides inscribed in a circle of radins R.

7. (*a*) Solve the equation $2(\sin^4\theta+\cos^4\theta)=1.$

(*b*) If $\sec(\phi+\alpha)+\sec(\phi-\alpha)=2\sec\phi$, prove that

$$\cos\phi=\sqrt{2}\cos\frac{\alpha}{2}.$$

(*c*) There is a flagstaff *h* feet high at the top of a cliff. From a point at the foot of the cliff the angles of elevation of the top and bottom of the flagstaff are α and β respectively. Find the height of the cliff.

Higher Secondary 1961 (J. & K. University)

Note :—Do questions worth 44 marks Complete questions are to be attempted].

1. (*a*) Prove that a radian is an angle of constant magnitude.

(*b*) Express 2·2 radian in the Sexagesimal and Centesimal Systems.

2. (*a*) Express all the circular functions of θ in terms of $\cos\theta$.

(*b*) Given that $\tan\theta = \frac{2}{5}$, when θ lies in the third quadrant, find the other circular functions of θ.

Or

(*b*) Eliminate θ from $a\cos\theta + b\sin\theta + c = 0$

$$a_1\cos\theta + b_1\sin\theta + c_1 = 0$$

3. (*a*) Prove that the logarithm of the product of two factors is equal to the sum of the logarithms of the factors.

(*b*) If $a^2 + b^2 = 7ab$, then $\log\left(\frac{a+b}{3}\right) =$
$\frac{1}{2}(\log a + \log b)$

4. (*a*) Solve the equation $5^{7-1x} = 2^{x+5}$, given that $\log 2 = \cdot 3010$.

(*b*) Given that Log $2 = \cdot 3010$, find the position of the first significant figure in 2^{-35}

5. (*a*) AD is the bisector of $\angle A$ of the $\triangle ABC$, meeting BC in D. Prove that

$$BD = \frac{a\sin C}{\sin C + \sin B}, \quad CD = \frac{a\sin B}{\sin C + \sin B}$$

(*b*) In a $\triangle ABC$, if $\frac{\cos A}{a} = \frac{\cos B}{b} = \frac{\cos C}{c}$,
Prove that the triangle is equilateral.

5. (*a*) A circle with radius R passes through the vertices A, B and C of the $\triangle$ ABC. Find the value of R.

(*b*) Prove that the $\triangle ABC = \dfrac{\text{Product of sides}}{4R}$

Or

(*b*) At a point 200 ft. from the base of a tower which stands on a horizontal plane, the angle of elevation of the top is 60°. Find the length of the tower.

ANSWERS.

Ex. 1. Page 8.

1. (*i*) first (*ii*) Second (*iii*) Third.
2. (*i*) 108° (*ii*) 126° (*iii*) 315°
3. (*i*) $\frac{\pi}{12}$ (*ii*) $\frac{3\pi}{10}$ (*iii*) $\frac{19\pi}{48}$ (*iv*) $\frac{47\pi}{75}$
4. $\frac{5\pi}{36}$, $\frac{\pi}{3}$, $\frac{19\pi}{36}$
6. $114^\circ\ 32'\ 43\frac{7}{11}''$.
7. $\frac{3}{5}$ radians or $34{\cdot}4^\circ$ nearly.
8. $5\frac{1}{2}$ ft.
9. $1\frac{9}{35}$ ft. 10. $\cdot238737$ miles nearly.
11. $34'\ 43''$ nearly. 12. $2{\cdot}86^\circ$

Ex. 2. Page 15.

25. $(ax-by)^2+(bx+ay)^2=(a^2+b^2)^2$ 26. $\frac{y^2}{b^2}-\frac{x^2}{a^2}=1$

Ex. 3. Page 21.

1. (*i*) positive (*ii*) negative (*iii*) negative (*iv*) positive (*v*) negative.
2. $\sin\theta=\pm\sqrt{1-\cos^2\theta}$, $\tan\theta=\pm\frac{\sqrt{1-\cos^2\theta}}{\cos\theta}$

$\cot\theta=\frac{\cos\theta}{\pm\sqrt{1-\cos^2\theta}}$, $\sec\theta=\frac{1}{\cos\theta}$

$\operatorname{cosec}\theta=\frac{1}{\pm\sqrt{1-\cos^2\theta}}$

3. $\sin\theta = \dfrac{\pm\sqrt{\sec^2\theta - 1}}{\sec\theta}$, $\cos\theta = \dfrac{1}{\sec\theta}$

$\tan\theta = \pm\sqrt{\sec^2\theta - 1}$, $\cot\theta = \dfrac{1}{\pm\sqrt{\sec^2\theta - 1}}$,

$\operatorname{cosec}\theta = \dfrac{\sec\theta}{\pm\sqrt{\sec^2\theta - 1}}$,

4. $\cot A = \dfrac{-3}{\sqrt{40}}$, $\operatorname{cosec} A = \dfrac{-7}{\sqrt{40}}$ 5. $1\dfrac{31}{65}$

6. (*i*) Yes (*ii*) no (*iii*) Yes (*iv*) no (*v*) Yes.

7. For one value only. 8. No.

11. (*i*) Second. (*ii*) Fourth.

12. First and third quadrants, $\sin\theta = \pm\frac{1}{2}$, $\cos\theta = \pm\dfrac{\sqrt{3}}{2}$.

$\cot\theta = \sqrt{3}$, $\sec\theta = \pm\dfrac{2}{\sqrt{3}}$, $\operatorname{cosec}\theta = \pm\sqrt{2}$

13. $\sin\theta = \pm\sqrt{\dfrac{5}{6}}$, $\cos\theta = \pm\dfrac{1}{\sqrt{6}}$, $\tan\theta = \pm\sqrt{5}$

$\cot\theta = \pm\dfrac{1}{\sqrt{5}}$, $\sec\theta = \pm\sqrt{6}$, $\operatorname{cosec}\theta = \pm\sqrt{\dfrac{6}{5}}$

14. $\pm\sqrt{5}$. 15. $14\frac{5}{11}$. 16. $-\frac{2}{37}$

Ex. 4. Page 28.

6. 1. 7. $\frac{1}{4}$ 8. Zero 9. Zero. 10. $\dfrac{\sqrt{3}-1}{2\sqrt{2}}$.

11. 30° 12. 60° 13. 90° or 30° 14. 30°
or $\cot^{-1}(-2\sqrt{3})$.

15. 45°, 15° 16. $52\frac{1}{2}$° and $7\frac{1}{2}$° 18. (*i*) $2\sqrt{\frac{2}{3}}$, (*ii*) $\frac{9}{8}$

Ex. 5. Page 42.

1. $\dfrac{-1}{\sqrt{2}}, -\dfrac{\sqrt{3}}{2}, -1, -2$ 7. (*i*) -1, (*ii*) 1.

13. (*i*) 30°, 330° (*ii*) 240°, 300°, (*iii*) 60°, 300°
(*iv*) 45°, 135° (*v*) 135°, 315°.

14. π.

Ex. 6. Page 62.

1. ·57, ·91 nearly 2. About $\pm 37°$, about $124°$
3. About $\pm 44°, \pm 136°$, 5. About $63°$
7. (*i*) $30°, 210°; 150°, 330°$; (*ii*) $45°, 225°; 135°, 315°$
10. $0°$, ·26 radians (*i. e.* $15°$). 12. $\frac{\pi}{4}, \frac{3\pi}{4}$
14. ·73 radians (*i. e.* $42°$)

Ex. 7. Page 68.

1. 50 ft. 2. $75\sqrt{3}$ yds. 3. 115·5 ft. 4. 42·3 ft.
5. 80 ft., $20\sqrt{3}$ ft. 6. 224 ft. 7. 100 $(\sqrt{3}+1)$ ft.
8. 100 $(3+\sqrt{3})$ ft. 9. $20\sqrt{3}$ ft. 10. 4·2 ft.
11. 819·6 ft. 12. 34 64 ft. and 20 ft.
13. $\frac{160}{\sqrt{3}}$ ft. 15. $30°$ 17. 1385 6 ft., 1385·6 ft.
18. 216·5 ft. 19. 42 ft. nearly. 20. $133\frac{1}{3}$.
21. height. $=50\sqrt{3}$, Breadth $=50$
22. ht. $=30\sqrt{3}$; the pt. is 30 ft. from one end.

Ex. 8. Page 80.

17. $\frac{2499}{2501}, \frac{-100}{2561}$ 28. $\frac{5}{16}$

Ex. 9. Page 85.

1. cos A cos B cos C+sin B sin C cos A
—sin C sin A cos B+sin A sin B cos C.
4. $2\cos\left(\theta-\frac{\pi}{6}\right)$; 2.
5. (*i*) $2\sin\left(\theta+\frac{\pi}{6}\right)$. (*ii*) $\sqrt{2}\sin\left(\theta+\frac{\pi}{4}\right)$

Ex. 10. Page 88.

1. $\frac{47}{49}$ 2. $\frac{-7}{25}, \frac{24}{25}$ 3. $\frac{120}{169}, \frac{-119}{169}$ 4. $\frac{8}{17}, \frac{15}{17}, \frac{8}{15}$.

5. $\frac{117}{125}, \frac{-44}{125}$ 22. a. 23. $16 \cos^5 A - 20 \cos^3 A + 5 \cos A$

Ex. 11. Page 92.

1. (*i*) $\sqrt{5}-1$, (*ii*) $\sqrt{5}+1$.

Ex. 13. Page 100.

1. $\frac{\sqrt{3}-1}{2\sqrt{2}}, -\frac{\sqrt{3}+1}{2\sqrt{2}}$. 2. $\frac{\sqrt{3}-1}{\sqrt{3}+1}$.

3. $\frac{\sqrt{3}-1}{2\sqrt{2}}, \frac{\sqrt{3}+1}{2\sqrt{2}}$

5. $\frac{\sqrt{2-\sqrt{2}}+\sqrt{2+\sqrt{2}}}{2\sqrt{2}}, \frac{\sqrt{2-\sqrt{2}}-\sqrt{2+\sqrt{2}}}{2\sqrt{2}}$

6. $\tan A = \frac{3}{7}$, $\sin \frac{A}{2} = \frac{3}{\sqrt{58}}$, $\cos \frac{A}{2} = \frac{7}{\sqrt{58}}$

8. $\frac{3\pi}{4}$ and $\frac{5\pi}{4}$ 9. $16 \cos^5\theta - 20 \cos^3\theta + 5 \cos \theta$

Ex. 14. Page 105.

1. $\sin 3A + \sin A$
2. $\cos 6x + \cos 4x$
3. $\cos 2A - \cos 4A$
4. $\sin 3x - \sin x$.
5. $2 \sin 4A \cos 2A$.
6. $-2 \sin 2A \sin A$.
7. $2 \cos \frac{7x}{2} \sin \frac{x}{2}$
8. $2 \sin \frac{3x}{2} \sin \frac{x}{2}$.
9. $-2 \cos \frac{7x}{2} \sin \frac{3x}{2}$.
10. $2 \cos 2x \cos x$

Ex. 16. Page 120.

1. (*i*) $n\pi + (-1)^n \frac{\pi}{3}$ (*ii*) $2n\pi \pm \frac{\pi}{4}$ (*iii*) $n\pi + \frac{\pi}{3}$

(iv) $2n\pi \pm \frac{2\pi}{3}$ (v) $2n\pi \pm \frac{\pi}{6}$ (vi) $n\pi + (-1)^n \frac{\pi}{4}$

(vii) $n\pi \pm \frac{\pi}{3}$ (viii) $n\pi \pm \frac{\pi}{3}$ (ix) $n\pi \pm \frac{\pi}{6}$

2. (i) $2n\pi + \frac{\pi}{4}$ (ii) $2n\pi \pm \frac{7\pi}{6}$ (iii) $2n\pi + \frac{5\pi}{6}$

3 $n\pi + (-)^n \frac{\pi}{6}$ or $n\pi - (-1)^n \frac{\pi}{2}$

4. $2n\pi \pm \frac{\pi}{3}$

5. $2n\pi \pm \frac{5\pi}{6}$ 6. $2n\pi$ or $2n\pi \pm \frac{2\pi}{3}$

7. $2n\pi \pm \frac{\pi}{3}$ 8 $n\pi + (-1)^n \frac{\pi}{6}$

9. $n\pi \pm \frac{\pi}{4}$ 10 $2n\pi \pm \frac{\pi}{3}$

11. $n\pi + \frac{\pi}{6}$ or $n\pi - \frac{\pi}{3}$ 12. $n\pi \pm \frac{\pi}{4}$

13. $n\pi \pm \alpha$ where $\alpha = \cos^{-1}\sqrt{\frac{c-b}{a-b}}$

14. $n\pi \pm \frac{\pi}{4}$ 15. $x = \frac{6m-4n}{5}\pi \pm \frac{\pi}{10} \pm \frac{2\pi}{15}$

$$y = \frac{6n-4m}{5}\pi \pm \frac{\pi}{5} \pm \frac{\pi}{3}$$

Ex. 17. Page 124.

1. $2n\pi \pm \frac{\pi}{4} + \frac{\pi}{6}$ 2. $2n\pi + \frac{\pi}{3}$ 3. $2n\pi$ or $2n\pi + \frac{\pi}{2}$

4. $2n\pi + \frac{\pi}{4}$ 5. $n\pi + (-1)^n \frac{\pi}{6} + \frac{\pi}{4}$ 6. $2n\pi - \frac{\pi}{4}$

7. $2n\pi + \frac{2\pi}{3}, 2n\pi$ 8. $\frac{p\pi}{m-(-1)^p n}$

9. $\dfrac{n\pi}{9-(-1)^n(10)}$ 10. $\dfrac{\pi}{20}(4n+1)$ or $(4n-1)\dfrac{\pi}{16}$

11. $\dfrac{\pi}{7}(n+\frac{1}{2})$ 12. $\dfrac{p\pi}{m-n}$

13. $(2n+1)\dfrac{\pi}{6}$ or $n\pi+(-1)^n\dfrac{\pi}{6}$

14. $\dfrac{n\pi}{2}$ or $2n\pi\pm\dfrac{2\pi}{3}$ 15. $\dfrac{2n\pi}{3}$ or $n\pi+\dfrac{\pi}{4}$ or $2n\pi-\dfrac{\pi}{2}$

16. $(2n+1)\dfrac{\pi}{2}$ or $(2n+1)\dfrac{\pi}{4}$ or $(2n+1)\dfrac{\pi}{8}$

17. $n\pi\pm\dfrac{\pi}{4}$, $2n\pi\pm\dfrac{2\pi}{3}$

18. $n\pi$ or $n\pi\pm\dfrac{\pi}{3}$ or $n\pi\pm\tan^{-1}\dfrac{1}{\sqrt{2}}$

19. $\dfrac{2n\pi}{3}\pm\dfrac{\pi}{6}$ 20. $2n\pi$ or $2n\pi+\dfrac{\pi}{2}$ 21. $2n\pi$ or $2n\pi+\dfrac{\pi}{2}$

22. $2n\pi\pm\dfrac{\pi}{3}$ or $2n\pi\pm\dfrac{2\pi}{3}$ 23. $\dfrac{n\pi}{2}+(-1)^n\left(\pm\dfrac{\pi}{4}\right)$

24. $\dfrac{n\pi}{4}$ or $\dfrac{2n\pi}{3}+\dfrac{\pi}{9}$

Ex. 20. Page 141

20. $\tan\dfrac{C}{2}=\frac{2}{5}$

Ex. 21. Page 144.

1. $10\sqrt{3}$ 2. 84 3. 108 4. 18 5. $36(3-\sqrt{3})$

Ex. 22. Page 156.

18. $\dfrac{15''}{\sqrt{11}}$ 19. $\dfrac{3\pi}{2}$ 20. 78·56 ft. nearly

21. 856·6 sq. units 22. 169·2 28. 12, 16, 20.

Ex 23. Page 167.

1. (*a*) (*i*) 3 (*ii*) 5 (*iii*) —1 (*iv*) —3 (*v*) —6.
   (*b*) (*i*) 8 (*ii*) $\frac{7}{5}$ (*iii*) $\frac{3}{2}$
2. 9317, 2·9317, $\bar{3}$·9317
4. (*i*) 8060 (*ii*) $\bar{3}$·3980 (*iii*) $\bar{3}$·5820
5. 30, the 16th. 6. 7 digits 7. (*i*) 1·59 (*ii*) 3·32
10. (*i*) —·97 nearly (*ii*) 28·48 nearly.

Ex. 24 Page 171.

1. (*i*) 1·6621 (*ii*) 2·9671 (*iii*) ·2157
2. (*i*) 3·142 (*ii*) 123·9 (*iii*) ·02709
3. (*i*) 3·892 (*ii*) ·0009342 (*iii*) 49·84 nearly (*iv*) ·03058
4. 3·033 5. 2·6

Ex. 25. Page 177.

1. (*i*) ·7911 (*ii*) ·8407 (*iii*) —2·9266 (*v*) ·6569
2. (*i*) 1·8394 (*ii*) 1·4793
3. (*i*) 16° 3′ (*ii*) 43° 16′ 4. (*i*) 29° 37′ (*ii*) 21° 5′
5. $\bar{1}$·6662. $\bar{1}$·6663
6. 23° 18′ 40″ 7. ·6506 8. 9·6356, 9·9980
9. 1·46 (*i*) —·2540, —·7604 (*ii*) 67° 23′
12. $2n\pi+306°$ 52′ nearly

Ex. 26. Page 180.

1. B = 40° 41′, c = 41·28, A = 49° 19′
2. A = 16° 46′, B = 73° 14′, b = 788·3
3. B = 82° 17′, b = 12·77, c = 12·89
4. a = 16·44, B = 56° 38′, b = 24·98
5. 263.77

6. $a=9{\cdot}367$, $A=43^\circ\ 56'$, $B=46^\circ\ 4'$
7. $a=5{\cdot}126$, $c=8{\cdot}170$, $B=51^\circ\ 8'$
8. $a=12{\cdot}88$, $c=29{\cdot}91$, $A=25^\circ\ 30'$

Ex. 27. Page 183.

1. $A=49^\circ\ 28'$, $B=58^\circ\ 46'$, $C=71^\circ\ 46'$
2. $A=76^\circ\ 6'$, $B=73^\circ\ 54'$, $C=30^\circ$
3. $A=87^\circ\ 20'$, $B=30^\circ\ 24'$, $C=62^\circ\ 16'$
4. $A=37^\circ\ 30'$, $B=53^\circ\ 32'$, $C=88^\circ\ 58'$
5. $C=132^\circ\ 35'$
6. $A=60^\circ\ 10'$
7. (*i*) $132^\circ\ 34'$ (*ii*) $130^\circ\ 42'\ 40''$ 8. $73^\circ\ 23'\ 54''$
9. $A=33^\circ\ 40'$, $B=101^\circ\ 56'\ 48''$, $C=44^\circ\ 23'\ 12''$
11. $104^\circ\ 29'$ to the nearest minute.
12. $A=60^\circ\ 10'$, $B=28^\circ\ 8'$, $C=91^\circ\ 42'$

Ex. 28 Page 187.

1. $B=120^\circ$, $C=30^\circ$, $a=1$.
2. $A=105^\circ$, $B=15^\circ$, $c=\sqrt{6}$.
3. $B=106^\circ\ 16'$, $C=36^\circ\ 52'$, $a=5$
4. $B=97^\circ\ 30'$, $C=35^\circ\ 30'$, $a=18{\cdot}51$
5. $A=37^\circ\ 18'$, $B=91^\circ\ 54'$, $c=27{\cdot}33$
6. $105^\circ\ 48'$, $32^\circ\ 32'$ nearly.
7. $70^\circ\ 53'\ 36''$, $49^\circ\ 6'\ 24''$,
8. $70^\circ\ 53'\ 37''$, $49^\circ\ 6'\ 23''$.
9. $51^\circ\ 12'\ 26''$. 10. $70^\circ\ 32'\ 46''$ and $34^\circ\ 27'\ 14''$.
11. $B=15^\circ\ 6'\ 20''$. $C=127^\circ\ 53'\ 40''$

Ex. 29. Page 189.

1. $C=47^\circ$, $b=123{\cdot}2$, $c=112{\cdot}8$

2. $A=59° 30'$, $b=61·51$, $c=32·51$
3. $A=42° 54'$, $b=25·06$, $c=26·54$
4. $C=65° 45'$, $b=22·66$, $c=21·63$
5. $C=42° 54'$, $a=663·4$, $b=624$,
6. 20·98.

Ex. 30. Page 194.

1. (*i*) $C_1=58° 57'$, $A_1=87° 48'$, $a_1=29·16$
   (*ii*) $C_2=121° 3'$, $A_2=25° 42'$, $a_2=12·64$,
2. No solution.
3. $B_1=51° 20'$, $C_1=98° 19'$ $c_1=21·54$.
   $B_2=128° 40'$, $C_2=20° 50'$, $c_2=7·796$,
4. $B=61°24'$. $C=48°21'$, $b=495.8$.
5. $B=25° 30'$, $C=82° 15'$, $c=190$.
6. (*i*) $A_1=49° 37'$, $B_1=87° 56'$, $b_1=1800$
   (*ii*) $A_2=130° 23'$, $B_2=7° 10'$, $b_2=1348$
7. 39° 35′ 10″, 28° 20′ 50″
8. (*i*) No, $\because c \sin A=a$ (*ii*) No, $\because$ a not less than c.
9. $B_1=48° 35' 25''$, $B_2=131° 24 35''$.
   $C_1=101° 24' 35''$, $C_2=18° 35' 25''$.
10. 70° 0′ 56″ or 109° 59′ 4″.

Ex. 31. Page 199.

5. one two, none.

Ex. 32. Page 204.

1. 69·23 ft. 2. 88·16 ft. 3. 200 ft., 26° 34′ nearly
5. 1·366 miles 6. 200 ft. 7. 175·9 ft. 8. 49·06 ft.
9. $50\sqrt{2}$ yds. 10. $h \tan \alpha \cot \beta$ 11. 5362 ft.

12. 18·87 ft., 32·68 ft. 15. $100\sqrt[3]{2}$ 16. 273·2 ft

18. 183 ft. 19. 1·43 miles

20. 163 ft. 21. 80·46 24. 273·2 ft. 25. 2 yds.

Ex. 33. Page 213.

1. 6·1554 in., 6·472 in., 123·108 sq. in.

2. $4-2\sqrt{2}$ 6. $n=6$.

Ex. 34. Page 220.

1. 42° 12′, 18·42, 230·25 sq. in., 20·35 sq. in

6. (i) $\frac{\pi}{64800}$ nearly (ii) ·01745 7. 6 sq. ft.

9. 24.

K. U. 1955.

1. (b) $\frac{11}{60}$ radians ; 10° 30′ (c) $\frac{-2}{37}$

4. (b) (i) $n\pi \pm \frac{\pi}{4}$

(ii) $2n\pi \pm \frac{\pi}{3} + \frac{\pi}{6}$

6. (a) 3605·712 ft.

(b) 130° 42′ 40″.

$\left[\textbf{Hint.}\ \text{Use } \tan\frac{A}{2}\right]$

K. U. 1956.

1. (b) $\operatorname{cosec} A = \pm\sqrt{5}$.

4. (b) (i) $\theta = n\pi + (-1)^n \frac{\pi}{6} + \frac{\pi}{4}$

(ii) $\theta = n\pi \pm \frac{\pi}{6}$

6. (*a*) $50\sqrt{3}$ yds. and 50 yds.
   (*b*) only one solution.
   (*c*) $C=36° \, 49'$, $A=100° \, 44'$
   $a=59.69$.

K. U. 1957.

2. (*c*) $30\sqrt{3}$; 30 ft. from one end.

5. (*c*) $\theta=\dfrac{n\pi}{4-(-1)^n}$

6. (*a*) $B=25° \, 30'$, $C=82° \, 15'$, $c=190$.

K. U. 1958.

1. (*b*) $1\frac{9}{25}$ ft. (*c*) $60°$; $66\frac{2}{3}°$.
3. (*a*) $680\,(3\pm\sqrt{3})$. (*b*) $45°$, $135°$.
5. (*a*) $\frac{1}{2}n\pi$, $2r\pi\pm\dfrac{2\pi}{3}$.
6. (*a*) 7. (*b*) $70° \, 0' \, 56''$ or $109° \, 59' \, 4''$

K. U. 1959

2. (*b*) $2n\pi\pm\dfrac{\pi}{3}$

K. U. 1960

1. (*b*) $133\frac{1}{3}$ ft. 3. (*b*) $\dfrac{n\pi}{4}$ or $\dfrac{2n\pi}{3}+\dfrac{\pi}{9}$
5. $B=120°$, $C=30°$, $a=1$.

K. U. 1961

3. (*b*) $\dfrac{n\pi}{3}$ or $n\pi\pm\dfrac{\pi}{6}$ 4. (a) thirty

Dehli Higher Secondary 1957

1. (*b*) 35 ft. 3. (*b*) r cosec $\frac{\alpha}{2}$ sin β

5. (*b*) $n\pi \pm \frac{\pi}{6}$ 6. (*a*) $a^2(4-\pi)$

6. (*b*) 2·5298233 nearly.

D. H. S. 1958

1. (*b*) $\frac{2\pi}{9}, -\frac{\pi}{3}, \frac{4\pi}{9}$. 3. (*a*) $\frac{\pi}{3}, \frac{3\pi}{3}; \frac{2\pi}{3}, \frac{5\pi}{3}$

5, (*b*) $2n\pi \pm \frac{\pi}{3}$ 6. (*b*) 120° 6. (*c*) 24 sq. ft.

D. H. S. 1959

1. (*a*) 38° 30′. 2. (*b*) −1.

4. (*b*) a=27·06, B=62° 38′ nearly, C=79° 6′ nearly.

7. (*a*) $\frac{n\pi}{2} \pm \frac{\pi}{4}$ 7. (*c*) $\frac{h \cos \alpha \sin \beta}{\sin(\alpha-\beta)}$

Higher Secondary 1961 (J. & K. U.)

1. (*b*) 126°, 140^g 2. (*b*) $\sin\theta=\frac{2}{\sqrt{29}}$,

$\cos\theta=\frac{5}{\sqrt{29}}$, $\cot\theta=\frac{5}{2}$, $\cos\theta=\frac{\sqrt{29}}{9}$,

$\sec\theta=\frac{\sqrt{29}}{5}$

or

2. (*b*) $(bc_1-b_1c)^2+(ca_1-c_1a)^2=(ab_1-a_1b)^3$

4. (*a*) $x=1\frac{291}{3097}$ (*b*) (10th.

5. (*b*) $200\sqrt{3}$ ft.

TABLES OF LOGARITHMS

LOGARITHMS

	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9
10	0000	0043	0086	0128	0170						5	9	13	17	21	26	30	34	38
						0212	0253	0294	0334	0374	4	8	12	16	20	24	28	32	36
11	0414	0453	0492	0531	0569						4	8	12	16	20	23	27	31	35
						0607	0645	0682	0719	0755	4	7	11	15	18	22	26	29	33
12	0792	0828	0864	0899	0934						3	7	11	14	18	21	25	28	32
						0969	1004	1038	1072	1106	3	7	10	14	17	20	24	27	31
13	1139	1173	1206	1239	1271						3	6	10	13	16	19	23	26	29
						1303	1335	1367	1399	1430	3	7	10	13	16	19	22	25	29
14	1461	1492	1523	1553	1584						3	6	9	12	15	19	22	25	28
						1614	1644	1673	1703	1732	3	6	9	12	14	17	20	23	26
15	1761	1790	1818	1847	1875						3	6	9	11	14	17	20	23	26
						1903	1931	1959	1987	2014	3	6	8	11	14	17	19	22	25
16	2041	2068	2095	2122	2148						3	6	8	11	14	16	19	22	24
						2175	2201	2227	2253	2279	3	5	8	10	13	16	18	21	23
17	2304	2330	2355	2380	2405						3	5	8	10	13	15	18	20	23
						2430	2455	2480	2504	2529	3	5	8	10	12	15	17	20	22
18	2553	2577	2601	2625	2648						2	5	7	9	12	14	17	19	21
						2672	2695	2718	2742	2765	2	4	7	9	11	14	16	18	21
19	2788	2810	2833	2856	2878						2	4	7	9	11	13	16	18	20
						2900	2923	2945	2967	2989	2	4	6	8	11	13	15	17	19
20	3010	3032	3054	3075	3096	3118	3139	3160	3181	3201	2	4	6	8	11	13	15	17	19
21	3222	3243	3263	3284	3304	3324	3345	3365	3385	3404	2	4	6	8	10	12	14	16	18
22	3424	3444	3464	3483	3502	3522	3541	3560	3579	3598	2	4	6	8	10	12	14	15	17
23	3617	3636	3655	3674	3692	3711	3729	3747	3766	3784	2	4	6	7	9	11	13	15	17
24	3802	3820	3838	3856	3874	3892	3909	3927	3945	3962	2	4	5	7	9	11	12	14	16
25	3979	3997	4014	4031	4048	4065	4082	4099	4116	4133	2	3	5	7	9	10	12	14	15
26	4150	4166	4183	4200	4216	4232	4249	4265	4281	4298	2	3	5	7	8	10	11	13	15
27	4314	4330	4346	4362	4378	4393	4409	4425	4440	4456	2	3	5	6	8	9	11	13	14
28	4472	4487	4502	4518	4533	4548	4564	4579	4594	4609	2	3	5	6	8	9	11	12	14
29	4624	4639	4654	4669	4683	4698	4713	4728	4742	4757	1	3	4	6	7	9	10	12	13
30	4771	4786	4800	4814	4829	4843	4857	4871	4886	4900	1	3	4	6	7	9	10	11	13
31	4914	4928	4942	4955	4969	4983	4997	5011	5024	5038	1	3	4	6	7	8	10	11	12
32	5051	5065	5079	5092	5105	5119	5132	5145	5159	5172	1	3	4	5	7	8	9	11	12
33	5185	5198	5211	5224	5237	5250	5263	5276	5289	5302	1	3	4	5	6	8	9	10	12
34	5315	5328	5340	5353	5366	5378	5391	5403	5416	5428	1	3	4	5	6	8	9	10	11
35	5441	5453	5465	5478	5490	5502	5514	5527	5539	5551	1	2	4	5	6	7	9	10	11
36	5563	5575	5587	5599	5611	5623	5635	5647	5658	5670	1	2	4	5	6	7	8	10	11
37	5682	5694	5705	5717	5729	5740	5752	5763	5775	5786	1	2	3	5	6	7	8	9	10
38	5798	5809	5821	5832	5843	5855	5866	5877	5888	5899	1	2	3	5	6	7	8	9	10
39	5911	5922	5933	5944	5955	5966	5977	5988	5999	6010	1	2	3	4	5	7	8	9	10
40	6021	6031	6042	6053	6064	6075	6085	6096	6107	6117	1	2	3	4	5	6	8	9	10
41	6128	6138	6149	6160	6170	6180	6191	6201	6212	6222	1	2	3	4	5	6	7	8	9
42	6232	6243	6253	6263	6274	6284	6294	6304	6314	6325	1	2	3	4	5	6	7	8	9
43	6335	6345	6355	6365	6375	6385	6395	6405	6415	6425	1	2	3	4	5	6	7	8	9
44	6435	6444	6454	6464	6474	6484	6493	6503	6513	6522	1	2	3	4	5	6	7	8	9
45	6532	6542	6551	6561	6571	6580	6590	6599	6609	6618	1	2	3	4	5	6	7	8	9
46	6628	6637	6646	6656	6665	6675	6684	6693	6702	6712	1	2	3	4	5	6	7	7	8
47	6721	6730	6739	6749	6758	6767	6776	6785	6794	6803	1	2	3	4	5	5	6	7	8
48	6812	6821	6830	6839	6848	6857	6866	6875	6884	6893	1	2	3	4	4	5	6	7	8
49	6902	6911	6920	6928	6937	6946	6955	6964	6972	6981	1	2	3	4	4	5	6	7	8

	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9
50	6990	6998	7007	7016	7024	7033	7042	7050	7059	7067	1	2	3	3	4	5	6	7	8
51	7076	7084	7093	7101	7110	7118	7126	7135	7143	7152	1	2	3	3	4	5	6	7	8
52	7160	7168	7177	7185	7193	7202	7210	7218	7226	7235	1	2	2	3	4	5	6	7	7
53	7243	7251	7259	7267	7275	7284	7292	7300	7308	7316	1	2	2	3	4	5	6	6	7
54	7324	7332	7340	7348	7356	7364	7372	7380	7388	7396	1	2	2	3	4	5	6	6	7
55	7404	7412	7419	7427	7435	7443	7451	7459	7466	7474	1	2	2	3	4	5	5	6	7
56	7482	7490	7497	7505	7513	7520	7528	7536	7543	7551	1	2	2	3	4	5	5	6	7
57	7559	7566	7574	7582	7589	7597	7604	7612	7619	7627	1	2	2	3	4	5	5	6	7
58	7634	7642	7649	7657	7664	7672	7679	7686	7694	7701	1	1	2	3	4	4	5	6	7
59	7709	7716	7723	7731	7738	7745	7752	7760	7767	7774	1	1	2	3	4	4	5	6	7
60	7782	7789	7796	7803	7810	7818	7825	7832	7839	7846	1	1	2	3	4	4	5	6	6
61	7853	7860	7868	7875	7882	7889	7896	7903	7910	7917	1	1	2	3	4	4	5	6	6
62	7924	7931	7938	7945	7952	7959	7966	7973	7980	7987	1	1	2	3	3	4	5	6	6
63	7993	8000	8007	8014	8021	8028	8035	8041	8048	8055	1	1	2	3	3	4	5	5	6
64	8062	8069	8075	8082	8089	8096	8102	8109	8116	8122	1	1	2	3	3	4	5	5	6
65	8129	8136	8142	8149	8156	8162	8169	8176	8182	8189	1	1	2	3	3	4	5	5	6
66	8195	8202	8209	8215	8222	8228	8235	8241	8248	8254	1	1	2	3	3	4	5	5	6
67	8261	8267	8274	8280	8287	8293	8299	8306	8312	8319	1	1	2	3	3	4	5	5	6
68	8325	8331	8338	8344	8351	8357	8363	8370	8376	8382	1	1	2	3	3	4	4	5	6
69	8388	8395	8401	8407	8414	8420	8426	8432	8439	8445	1	1	2	2	3	4	4	5	6
70	8451	8457	8463	8470	8476	8482	8488	8494	8500	8506	1	1	2	2	3	4	4	5	6
71	8513	8519	8525	8531	8537	8543	8549	8555	8561	8567	1	1	2	2	3	4	4	5	5
72	8573	8579	8585	8591	8597	8603	8609	8615	8621	8627	1	1	2	2	3	4	4	5	5
73	8633	8639	8645	8651	8657	8663	8669	8675	8681	8686	1	1	2	2	3	4	4	5	5
74	8692	8698	8704	8710	8716	8722	8727	8733	8739	8745	1	1	2	2	3	4	4	5	5
75	8751	8756	8762	8768	8774	8779	8785	8791	8797	8802	1	1	2	2	3	3	4	5	5
76	8808	8814	8820	8825	8831	8837	8842	8848	8854	8859	1	1	2	2	3	3	4	5	5
77	8865	8871	8876	8882	8887	8893	8899	8904	8910	8915	1	1	2	2	3	3	4	4	5
78	8921	8927	8932	8938	8943	8949	8954	8960	8965	8971	1	1	2	2	3	3	4	4	5
79	8976	8982	8987	8993	8998	9004	9009	9015	9020	9025	1	1	2	2	3	3	4	4	5
80	9031	9036	9042	9047	9053	9058	9063	9069	9074	9079	1	1	2	2	3	3	4	4	5
81	9085	9090	9096	9101	9106	9112	9117	9122	9128	9133	1	1	2	2	3	3	4	4	5
82	9138	9143	9149	9154	9159	9165	9170	9175	9180	9186	1	1	2	2	3	3	4	4	5
83	9191	9196	9201	9206	9212	9217	9222	9227	9232	9238	1	1	2	2	3	3	4	4	5
84	9243	9248	9253	9258	9263	9269	9274	9279	9284	9289	1	1	2	2	3	3	4	4	5
85	9294	9299	9304	9309	9315	9320	9325	9330	9335	9340	1	1	2	2	3	3	4	4	5
86	9345	9350	9355	9360	9365	9370	9375	9380	9385	9390	1	1	2	2	3	3	4	4	5
87	9395	9400	9405	9410	9415	9420	9425	9430	9435	9440	0	1	1	2	2	3	3	4	4
88	9445	9450	9455	9460	9465	9469	9474	9479	9484	9489	0	1	1	2	2	3	3	4	4
89	9494	9499	9504	9509	9513	9518	9523	9528	9533	9538	0	1	1	2	2	3	3	4	4
90	9542	9547	9552	9557	9562	9566	9571	9576	9581	9586	0	1	1	2	2	3	3	4	4
91	9590	9595	9600	9605	9609	9614	9619	9624	9628	9633	0	1	1	2	2	3	3	4	4
92	9638	9643	9647	9652	9657	9661	9666	9671	9675	9680	0	1	1	2	2	3	3	4	4
93	9685	9689	9694	9699	9703	9708	9713	9717	9722	9727	0	1	1	2	2	3	3	4	4
94	9731	9736	9741	9745	9750	9754	9759	9763	9768	9773	0	1	1	2	2	3	3	4	4
95	9777	9782	9786	9791	9795	9800	9805	9809	9814	9818	0	1	1	2	2	3	3	4	4
96	9823	9827	9832	9836	9841	9845	9850	9854	9859	9863	0	1	1	2	2	3	3	4	4
97	9868	9872	9877	9881	9886	9890	9894	9899	9903	9908	0	1	1	2	2	3	3	4	4
98	9912	9917	9921	9926	9930	9934	9939	9943	9948	9952	0	1	1	2	2	3	3	4	4
99	9956	9961	9965	9969	9974	9978	9983	9987	9991	9996	0	1	1	2	2	3	3	3	4

ANTILOGARITHMS

	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9
·50	3162	3170	3177	3184	3192	3199	3206	3214	3221	3228	1	1	2	3	4	4	5	6	7
·51	3236	3243	3251	3258	3266	3273	3281	3289	3296	3304	1	2	2	3	4	5	5	6	7
·52	3311	3319	3327	3334	3342	3350	3357	3365	3373	3381	1	2	2	3	4	5	5	6	7
·53	3388	3396	3404	3412	3420	3428	3436	3443	3451	3459	1	2	2	3	4	5	6	6	7
·54	3467	3475	3483	3491	3499	3508	3516	3524	3532	3540	1	2	2	3	4	5	6	6	7
·55	3548	3556	3565	3573	3581	3589	3597	3606	3614	3622	1	2	2	3	4	5	6	7	7
·56	3631	3639	3648	3656	3664	3673	3681	3690	3698	3707	1	2	3	3	4	5	6	7	8
·57	3715	3724	3733	3741	3750	3758	3767	3776	3784	3793	1	2	3	3	4	5	6	7	8
·58	3802	3811	3819	3828	3837	3846	3855	3864	3873	3882	1	2	3	4	4	5	6	7	8
·59	3890	3899	3908	3917	3926	3936	3945	3954	3963	3972	1	2	3	4	5	5	6	7	8
·60	3981	3990	3999	4009	4018	4027	4036	4046	4055	4064	1	2	3	4	5	6	6	7	8
·61	4074	4083	4093	4102	4111	4121	4130	4140	4150	4159	1	2	3	4	5	6	7	8	9
·62	4169	4178	4188	4198	4207	4217	4227	4236	4246	4256	1	2	3	4	5	6	7	8	9
·63	4266	4276	4285	4295	4305	4315	4325	4335	4345	4355	1	2	3	4	5	6	7	8	9
·64	4365	4375	4385	4395	4406	4416	4426	4436	4446	4457	1	2	3	4	5	6	7	8	9
·65	4467	4477	4487	4498	4508	4519	4529	4539	4550	4560	1	2	3	4	5	6	7	8	9
·66	4571	4581	4592	4603	4613	4624	4634	4645	4656	4667	1	2	3	4	5	6	7	9	10
·67	4677	4688	4699	4710	4721	4732	4742	4753	4764	4775	1	2	3	4	5	7	8	9	10
·68	4786	4797	4808	4819	4831	4842	4853	4864	4875	4887	1	2	3	4	6	7	8	9	10
·69	4898	4909	4920	4932	4943	4955	4966	4977	4989	5000	1	2	3	5	6	7	8	9	10
·70	5012	5023	5035	5047	5058	5070	5082	5093	5105	5117	1	2	4	5	6	7	8	9	11
·71	5129	5140	5152	5164	5176	5188	5200	5212	5224	5236	1	2	4	5	6	7	8	10	11
·72	5248	5260	5272	5284	5297	5309	5321	5333	5346	5358	1	2	4	5	6	7	9	10	11
·73	5370	5383	5395	5408	5420	5433	5445	5458	5470	5483	1	3	4	5	6	8	9	10	11
·74	5495	5508	5521	5534	5546	5559	5572	5585	5598	5610	1	3	4	5	6	8	9	10	12
·75	5623	5636	5649	5662	5675	5689	5702	5715	5728	5741	1	3	4	5	7	8	9	10	12
·76	5754	5768	5781	5794	5808	5821	5834	5848	5861	5875	1	3	4	5	7	8	9	11	12
·77	5888	5902	5916	5929	5943	5957	5970	5984	5998	6012	1	3	4	5	7	8	10	11	12
·78	6026	6039	6053	6067	6081	6095	6109	6124	6138	6152	1	3	4	6	7	8	10	11	13
·79	6166	6180	6194	6209	6223	6237	6252	6266	6281	6295	1	3	4	6	7	9	10	11	13
·80	6310	6324	6339	6353	6368	6383	6397	6412	6427	6442	1	3	4	6	7	9	10	12	13
·81	6457	6471	6486	6501	6516	6531	6546	6561	6577	6592	2	3	5	6	8	9	11	12	14
·82	6607	6622	6637	6653	6668	6683	6699	6714	6730	6745	2	3	5	6	8	9	11	12	14
·83	6761	6776	6792	6808	6823	6839	6855	6871	6887	6902	2	3	5	6	8	9	11	13	14
·84	6918	6934	6950	6966	6982	6998	7015	7031	7047	7063	2	3	5	6	8	10	11	13	15
·85	7079	7096	7112	7129	7145	7161	7178	7194	7211	7228	2	3	5	7	8	10	12	13	15
·86	7244	7261	7278	7295	7311	7328	7345	7362	7379	7396	2	3	5	7	8	10	12	13	15
·87	7413	7430	7447	7464	7482	7499	7516	7534	7551	7568	2	3	5	7	9	10	12	14	16
·88	7586	7603	7621	7638	7656	7674	7691	7709	7727	7745	2	4	5	7	9	11	12	14	16
·89	7762	7780	7798	7816	7834	7852	7870	7889	7907	7925	2	4	5	7	9	11	13	14	16
·90	7943	7962	7980	7998	8017	8035	8054	8072	8091	8110	2	4	6	7	9	11	13	15	17
·91	8128	8147	8166	8185	8204	8222	8241	8260	8279	8299	2	4	6	8	9	11	13	15	17
·92	8318	8337	8356	8375	8395	8414	8433	8453	8472	8492	2	4	6	8	10	12	14	15	17
·93	8511	8531	8551	8570	8590	8610	8630	8650	8670	8690	2	4	6	8	10	12	14	16	18
·94	8710	8730	8750	8770	8790	8810	8831	8851	8872	8892	2	4	6	8	10	12	14	16	18
·95	8913	8933	8954	8974	8995	9016	9036	9057	9078	9099	2	4	6	8	10	12	15	17	19
·96	9120	9141	9162	9183	9204	9226	9247	9268	9290	9311	2	4	6	8	11	13	15	17	19
·97	9333	9354	9376	9397	9419	9441	9462	9484	9506	9528	2	4	7	9	11	13	15	17	20
·98	9550	9572	9594	9616	9638	9661	9683	9705	9727	9750	2	4	7	9	11	13	16	18	20
·99	9772	9795	9817	9840	9863	9886	9908	9931	9954	9977	2	5	7	9	11	14	16	18	20

	0	1	2	3	4	5	6	7	8	9	123	456	789
·00	1000	1002	1005	1007	1009	1012	1014	1016	1019	1021	0 0 1	1 1 1	2 2 2
·01	1023	1026	1028	1030	1033	1035	1038	1040	1042	1045	0 0 1	1 1 1	2 2 2
·02	1047	1050	1052	1054	1057	1059	1062	1064	1067	1069	0 0 1	1 1 1	2 2 2
·03	1072	1074	1076	1079	1081	1084	1086	1089	1091	1094	0 0 1	1 1 1	2 2 2
·04	1096	1099	1102	1104	1107	1109	1112	1114	1117	1119	0 1 1	1 1 2	2 2 2
·05	1122	1125	1127	1130	1132	1135	1138	1140	1143	1146	0 1 1	1 1 2	2 2 2
·06	1148	1151	1153	1156	1159	1161	1164	1167	1169	1172	0 1 1	1 1 2	2 2 2
·07	1175	1178	1180	1183	1186	1189	1191	1194	1197	1199	0 1 1	1 1 2	2 2 2
·08	1202	1205	1208	1211	1213	1216	1219	1222	1225	1227	0 1 1	1 1 2	2 2 3
·09	1230	1233	1236	1239	1242	1245	1247	1250	1253	1256	0 1 1	1 1 2	2 2 3
·10	1259	1262	1265	1268	1271	1274	1276	1279	1282	1285	0 1 1	1 1 2	2 2 3
·11	1288	1291	1294	1297	1300	1303	1306	1309	1312	1315	0 1 1	1 2 2	2 2 3
·12	1318	1321	1324	1327	1330	1334	1337	1340	1343	1346	0 1 1	1 2 2	2 2 3
·13	1349	1352	1355	1358	1361	1365	1368	1371	1374	1377	0 1 1	1 2 2	2 3 3
·14	1380	1384	1387	1390	1393	1396	1400	1403	1406	1409	0 1 1	1 2 2	2 3 3
·15	1413	1416	1419	1422	1426	1429	1432	1435	1439	1442	0 1 1	1 2 2	2 3 3
·16	1445	1449	1452	1455	1459	1462	1466	1469	1472	1476	0 1 1	1 2 2	2 3 3
·17	1479	1483	1486	1489	1493	1496	1500	1503	1507	1510	0 1 1	1 2 2	2 3 3
·18	1514	1517	1521	1524	1528	1531	1535	1538	1542	1545	0 1 1	1 2 2	2 3 3
·19	1549	1552	1556	1560	1563	1567	1570	1574	1578	1581	0 1 1	1 2 2	3 3 3
·20	1585	1589	1592	1596	1600	1603	1607	1611	1614	1618	0 1 1	1 2 2	3 3 3
·21	1622	1626	1629	1633	1637	1641	1644	1648	1652	1656	0 1 1	2 2 2	3 3 3
·22	1660	1663	1667	1671	1675	1679	1683	1687	1690	1694	0 1 1	2 2 2	3 3 3
·23	1698	1702	1706	1710	1714	1718	1722	1726	1730	1734	0 1 1	2 2 2	3 3 4
·24	1738	1742	1746	1750	1754	1758	1762	1766	1770	1774	0 1 1	2 2 2	3 3 4
·25	1778	1782	1786	1791	1795	1799	1803	1807	1811	1816	0 1 1	2 2 2	3 3 4
·26	1820	1824	1828	1832	1837	1841	1845	1849	1854	1858	0 1 1	2 2 3	3 3 4
·27	1862	1866	1871	1875	1879	1884	1888	1892	1897	1901	0 1 1	2 2 3	3 3 4
·28	1905	1910	1914	1919	1923	1928	1932	1936	1941	1945	0 1 1	2 2 3	3 4 4
·29	1950	1954	1959	1963	1968	1972	1977	1982	1986	1991	0 1 1	2 2 3	3 4 4
·30	1995	2000	2004	2009	2014	2018	2023	2028	2032	2037	0 1 1	2 2 3	3 4 4
·31	2042	2046	2051	2056	2061	2065	2070	2075	2080	2084	0 1 1	2 2 3	3 4 4
·32	2089	2094	2099	2104	2109	2113	2118	2123	2128	2133	0 1 1	2 2 3	3 4 4
·33	2138	2143	2148	2153	2158	2163	2168	2173	2178	2183	0 1 1	2 2 3	3 4 4
·34	2188	2193	2198	2203	2208	2213	2218	2223	2228	2234	1 1 2	2 3 3	4 4 5
·35	2239	2244	2249	2254	2259	2265	2270	2275	2280	2286	1 1 2	2 3 3	4 4 5
·36	2291	2296	2301	2307	2312	2317	2323	2328	2333	2339	1 1 2	2 3 3	4 4 5
·37	2344	2350	2355	2360	2366	2371	2377	2382	2388	2393	1 1 2	2 3 3	4 4 5
·38	2399	2404	2410	2415	2421	2427	2432	2438	2443	2449	1 1 2	2 3 3	4 4 5
·39	2455	2460	2466	2472	2477	2483	2489	2495	2500	2506	1 1 2	2 3 3	4 5 5
·40	2512	2518	2523	2529	2535	2541	2547	2553	2559	2564	1 1 2	2 3 4	4 5 5
·41	2570	2576	2582	2588	2594	2600	2606	2612	2618	2624	1 1 2	2 3 4	4 5 5
·42	2630	2636	2642	2649	2655	2661	2667	2673	2679	2685	1 1 2	2 3 4	4 5 6
·43	2692	2698	2704	2710	2716	2723	2729	2735	2742	2748	1 1 2	3 3 4	4 5 6
·44	2754	2761	2767	2773	2780	2786	2793	2799	2805	2812	1 1 2	3 3 4	4 5 6
·45	2818	2825	2831	2838	2844	2851	2858	2864	2871	2877	1 1 2	3 3 4	5 5 6
·46	2884	2891	2897	2904	2911	2917	2924	2931	2938	2944	1 1 2	3 3 4	5 5 6
·47	2951	2958	2965	2972	2979	2985	2992	2999	3006	3013	1 1 2	3 3 4	5 5 6
·48	3020	3027	3034	3041	3048	3055	3062	3069	3076	3083	1 1 2	3 4 4	5 6 6
·49	3090	3097	3105	3112	3119	3126	3133	3141	3148	3155	1 1 2	3 4 4	5 6 6

NATURAL SINES

Degrees	0′ 0°·0	6′ 0°·1	12′ 0°·2	18′ 0°·3	24′ 0°·4	30′ 0°·5	36′ 0°·6	42′ 0°·7	48′ 0°·8	54′ 0°·9	Mean Differences 1	2	3	4	5
0	0000	0017	0035	0052	0070	0087	0105	0122	0140	0157	3	6	9	12	15
1	·0175	0192	0209	0227	0244	0262	0279	0297	0314	0332	3	6	9	12	15
2	·0349	0366	0384	0401	0419	0436	0454	0471	0488	0506	3	6	9	12	15
3	·0523	0541	0558	0576	0593	0610	0628	0645	0663	0680	3	6	9	12	15
4	·0698	0715	0732	0750	0767	0785	0802	0819	0837	0854	3	6	9	12	15
5	·0872	0889	0906	0924	0941	0958	0976	0993	1011	1028	3	6	9	12	14
6	·1045	1063	1080	1097	1115	1132	1149	1167	1184	1201	3	6	9	12	14
7	·1219	1236	1253	1271	1288	1305	1323	1340	1357	1374	3	6	9	12	14
8	·1392	1409	1426	1444	1461	1478	1495	1513	1530	1547	3	6	9	12	14
9	·1564	1582	1599	1616	1633	1650	1668	1685	1702	1719	3	6	9	12	14
10	·1736	1754	1771	1788	1805	1822	1840	1857	1874	1891	3	6	9	12	14
11	·1908	1925	1942	1959	1977	1994	2011	2028	2045	2062	3	6	9	11	14
12	·2079	2096	2113	2130	2147	2164	2181	2198	2215	2232	3	6	9	11	14
13	·2250	2267	2284	2300	2317	2334	2351	2368	2385	2402	3	6	8	11	14
14	·2419	2436	2453	2470	2487	2504	2521	2538	2554	2571	3	6	8	11	14
15	·2588	2605	2622	2639	2656	2672	2689	2706	2723	2740	3	6	8	11	14
16	·2756	2773	2790	2807	2823	2840	2857	2874	2890	2907	3	6	8	11	14
17	·2924	2940	2957	2974	2990	3007	3024	3040	3057	3074	3	6	8	11	14
18	·3090	3107	3123	3140	3156	3173	3190	3206	3223	3239	3	6	8	11	14
19	·3256	3272	3289	3305	3322	3338	3355	3371	3387	3404	3	5	8	11	14
20	·3420	3437	3453	3469	3486	3502	3518	3535	3551	3567	3	5	8	11	14
21	·3584	3600	3616	3633	3649	3665	3681	3697	3714	3730	3	5	8	11	14
22	·3746	3762	3778	3795	3811	3827	3843	3859	3875	3891	3	5	8	11	14
23	·3907	3923	3939	3955	3971	3987	4003	4019	4035	4051	3	5	8	11	14
24	·4067	4083	4099	4115	4131	4147	4163	4179	4195	4210	3	5	8	11	13
25	·4226	4242	4258	4274	4289	4305	4321	4337	4352	4368	3	5	8	11	13
26	·4384	4399	4415	4431	4446	4462	4478	4493	4509	4524	3	5	8	10	13
27	·4540	4555	4571	4586	4602	4617	4633	4648	4664	4679	3	5	8	10	13
28	·4695	4710	4726	4741	4756	4772	4787	4802	4818	4833	3	5	8	10	13
29	·4848	4863	4879	4894	4909	4924	4939	4955	4970	4985	3	5	8	10	13
30	·5000	5015	5030	5045	5060	5075	5090	5105	5120	5135	3	5	8	10	13
31	·5150	5165	5180	5195	5210	5225	5240	5255	5270	5284	2	5	7	10	12
32	·5299	5314	5329	5344	5358	5373	5388	5402	5417	5432	2	5	7	10	12
33	·5446	5461	5476	5490	5505	5519	5534	5548	5563	5577	2	5	7	10	12
34	·5592	5606	5621	5635	5650	5664	5678	5693	5707	5721	2	5	7	10	12
35	·5736	5750	5764	5779	5793	5807	5821	5835	5850	5864	2	5	7	10	12
36	·5878	5892	5906	5920	5934	5948	5962	5976	5990	6004	2	5	7	9	12
37	·6018	6032	6046	6060	6074	6088	6101	6115	6129	6143	2	5	7	9	12
38	·6157	6170	6184	6198	6211	6225	6239	6252	6266	6280	2	5	7	9	11
39	·6293	6307	6320	6334	6347	6361	6374	6388	6401	6414	2	4	7	9	11
40	·6428	6441	6455	6468	6481	6494	6508	6521	6534	6547	2	4	7	9	11
41	·6561	6574	6587	6600	6613	6626	6639	6652	6665	6678	2	4	7	9	11
42	·6691	6704	6717	6730	6743	6756	6769	6782	6794	6807	2	4	6	9	11
43	·6820	6833	6845	6858	6871	6884	6896	6909	6921	6934	2	4	6	8	11
44	·6947	6959	6972	6984	6997	7009	7022	7034	7046	7059	2	4	6	8	10

Degrees	0′ 0°·0	6′ 0°·1	12′ 0°·2	18′ 0°·3	24′ 0°·4	30′ 0°·5	36′ 0°·6	42′ 0°·7	48′ 0°·8	54′ 0°·9	Mean Differences 1	2	3	4	5
45	·7071	7083	7096	7108	7120	7133	7145	7157	7169	7181	2	4	6	8	10
46	·7193	7206	7218	7230	7242	7254	7266	7278	7290	7302	2	4	6	8	10
47	·7314	7325	7337	7349	7361	7373	7385	7396	7408	7420	2	4	6	8	10
48	·7431	7443	7455	7466	7478	7490	7501	7513	7524	7536	2	4	6	8	10
49	·7547	7558	7570	7581	7593	7604	7615	7627	7638	7649	2	4	6	8	9
50	·7660	7672	7683	7694	7705	7716	7727	7738	7749	7760	2	4	6	7	9
51	·7771	7782	7793	7804	7815	7826	7837	7848	7859	7869	2	4	5	7	9
52	·7880	7891	7902	7912	7923	7934	7944	7955	7965	7976	2	4	5	7	9
53	·7986	7997	8007	8018	8028	8039	8049	8059	8070	8080	2	3	5	7	9
54	·8090	8100	8111	8121	8131	8141	8151	8161	8171	8181	2	3	5	7	8
55	·8192	8202	8211	8221	8231	8241	8251	8261	8271	8281	2	3	5	7	8
56	·8290	8300	8310	8320	8329	8339	8348	8358	8368	8377	2	3	5	6	8
57	·8387	8396	8406	8415	8425	8434	8443	8453	8462	8471	2	3	5	6	8
58	·8480	8490	8499	8508	8517	8526	8536	8545	8554	8563	2	3	5	6	8
59	·8572	8581	8590	8599	8607	8616	8625	8634	8643	8652	1	3	4	6	7
60	·8660	8669	8678	8686	8695	8704	8712	8721	8729	8738	1	3	4	6	7
61	·8746	8755	8763	8771	8780	8788	8796	8805	8813	8821	1	3	4	6	7
62	·8829	8838	8846	8854	8862	8870	8878	8886	8894	8902	1	3	4	5	7
63	·8910	8918	8926	8934	8942	8949	8957	8965	8973	8980	1	3	4	5	6
64	·8988	8996	9003	9011	9018	9026	9033	9041	9048	9056	1	3	4	5	6
65	·9063	9070	9078	9085	9092	9100	9107	9114	9121	9128	1	2	4	5	6
66	·9135	9143	9150	9157	9164	9171	9178	9184	9191	9198	1	2	3	5	6
67	·9205	9212	9219	9225	9232	9239	9245	9252	9259	9265	1	2	3	4	6
68	·9272	9278	9285	9291	9298	9304	9311	9317	9323	9330	1	2	3	4	5
69	·9336	9342	9348	9354	9361	9367	9373	9379	9385	9391	1	2	3	4	5
70	·9397	9403	9409	9415	9421	9426	9432	9438	9444	9449	1	2	3	4	5
71	·9455	9461	9466	9472	9478	9483	9489	9494	9500	9505	1	2	3	4	5
72	·9511	9516	9521	9527	9532	9537	9542	9548	9553	9558	1	2	3	3	4
73	·9563	9568	9573	9578	9583	9588	9593	9598	9603	9608	1	2	2	3	4
74	·9613	9617	9622	9627	9632	9636	9641	9646	9650	9655	1	2	2	3	4
75	·9659	9664	9668	9673	9677	9681	9686	9690	9694	9699	1	1	2	3	4
76	·9703	9707	9711	9715	9720	9724	9728	9732	9736	9740	1	1	2	3	3
77	·9744	9748	9751	9755	9759	9763	9767	9770	9774	9778	1	1	2	3	3
78	·9781	9785	9789	9792	9796	9799	9803	9806	9810	9813	1	1	2	2	3
79	·9816	9820	9823	9826	9829	9833	9836	9839	9842	9845	1	1	2	2	3
80	·9848	9851	9854	9857	9860	9863	9866	9869	9871	9874	0	1	1	2	2
81	·9877	9880	9882	9885	9888	9890	9893	9895	9898	9900	0	1	1	2	2
82	·9903	9905	9907	9910	9912	9914	9917	9919	9921	9923	0	1	1	2	2
83	·9925	9928	9930	9932	9934	9936	9938	9940	9942	9943	0	1	1	1	2
84	·9945	9947	9949	9951	9952	9954	9956	9957	9959	9960	0	1	1	1	2
85	·9962	9963	9965	9966	9968	9969	9971	9972	9973	9974	0	0	1	1	1
86	·9976	9977	9978	9979	9980	9981	9982	9983	9984	9985	0	0	1	1	1
87	·9986	9987	9988	9989	9990	9990	9991	9992	9993	9993	0	0	0	1	1
88	·9994	9995	9995	9996	9996	9997	9997	9997	9998	9998	0	0	0	0	0
89	·9998	9999	9999	9999	9999	1·000	1·000	1·000	1·000	1·000	0	0	0	0	0
90	1·000														

NATURAL COSINES

[Numbers in difference columns to be subtracted, not added]

Degrees	0′ 0°·0	6′ 0°·1	12′ 0°·2	18′ 0°·3	24′ 0°·4	30′ 0°·5	36′ 0°·6	42′ 0°·7	48′ 0°·8	54′ 0°·9	Mean Differences 1	2	3	4	5
0	1·000	1·000	1·000	1·000	1·000	1·000	·9999	9999	9999	9999	0	0	0	0	0
1	·9998	9998	9998	9997	9997	9997	9996	9996	9995	9995	0	0	0	0	0
2	·9994	9993	9993	9992	9991	9990	9990	9989	9988	9987	0	0	0	1	1
3	·9986	9985	9984	9983	9982	9981	9980	9979	9978	9977	0	0	1	1	1
4	·9976	9974	9973	9972	9971	9969	9968	9966	9965	9963	0	0	1	1	1
5	·9962	9960	9959	9957	9956	9954	9952	9951	9949	9947	0	1	1	1	2
6	·9945	9943	9942	9940	9938	9936	9934	9932	9930	9928	0	1	1	1	2
7	·9925	9923	9921	9919	9917	9914	9912	9910	9907	9905	0	1	1	2	2
8	·9903	9900	9898	9895	9893	9890	9888	9885	9882	9880	0	1	1	2	2
9	·9877	9874	9871	9869	9866	9863	9860	9857	9854	9851	0	1	1	2	2
10	·9848	9845	9842	9839	9836	9833	9829	9826	9823	9820	1	1	2	2	3
11	·9816	9813	9810	9806	9803	9799	9796	9792	9789	9785	1	1	2	2	3
12	·9781	9778	9774	9770	9767	9763	9759	9755	9751	9748	1	1	2	3	3
13	·9744	9740	9736	9732	9728	9724	9720	9715	9711	9707	1	1	2	3	3
14	·9703	9699	9694	9690	9686	9681	9677	9673	9668	9664	1	1	2	3	4
15	·9659	9655	9650	9646	9641	9636	9632	9627	9622	9617	1	2	2	3	4
16	·9613	9608	9603	9598	9593	9588	9583	9578	9573	9568	1	2	2	3	4
17	·9563	9558	9553	9548	9542	9537	9532	9527	9521	9516	1	2	3	3	4
18	·9511	9505	9500	9494	9489	9483	9478	9472	9466	9461	1	2	3	4	5
19	·9455	9449	9444	9438	9432	9426	9421	9415	9409	9403	1	2	3	4	5
20	·9397	9391	9385	9379	9373	9367	9361	9354	9348	9342	1	2	3	4	5
21	·9336	9330	9323	9317	9311	9304	9298	9291	9285	9278	1	2	3	4	5
22	·9272	9265	9259	9252	9245	9239	9232	9225	9219	9212	1	2	3	4	6
23	·9205	9198	9191	9184	9178	9171	9164	9157	9150	9143	1	2	3	5	6
24	·9135	9128	9121	9114	9107	9100	9092	9085	9078	9070	1	2	4	5	6
25	·9063	9056	9048	9041	9033	9026	9018	9011	9003	8996	1	3	4	5	6
26	·8988	8980	8973	8965	8957	8949	8942	8934	8926	8918	1	3	4	5	6
27	·8910	8902	8894	8886	8878	8870	8862	8854	8846	8838	1	3	4	5	7
28	·8829	8821	8813	8805	8796	8788	8780	8771	8763	8755	1	3	4	6	7
29	·8746	8738	8729	8721	8712	8704	8695	8686	8678	8669	1	3	4	6	7
30	·8660	8652	8643	8634	8625	8616	8607	8599	8590	8581	1	3	4	6	7
31	·8572	8563	8554	8545	8536	8526	8517	8508	8499	8490	2	3	5	6	8
32	·8480	8471	8462	8453	8443	8434	8425	8415	8406	8396	2	3	5	6	8
33	·8387	8377	8368	8358	8348	8339	8329	8320	8310	8300	2	3	5	6	8
34	·8290	8281	8271	8261	8251	8241	8231	8221	8211	8202	2	3	5	7	8
35	·8192	8181	8171	8161	8151	8141	8131	8121	8111	8100	2	3	5	7	8
36	·8090	8080	8070	8059	8049	8039	8028	8018	8007	7997	2	3	5	7	9
37	·7986	7976	7965	7955	7944	7934	7923	7912	7902	7891	2	4	5	7	9
38	·7880	7869	7859	7848	7837	7826	7815	7804	7793	7782	2	4	5	7	9
39	·7771	7760	7749	7738	7727	7716	7705	7694	7683	7672	2	4	6	7	9
40	·7660	7649	7638	7627	7615	7604	7593	7581	7570	7559	2	4	6	8	9
41	·7547	7536	7524	7513	7501	7490	7478	7466	7455	7443	2	4	6	8	10
42	·7431	7420	7408	7396	7385	7373	7361	7349	7337	7325	2	4	6	8	10
43	·7314	7302	7290	7278	7266	7254	7242	7230	7218	7206	2	4	6	8	10
44	·7193	7181	7169	7157	7145	7133	7120	7108	7096	7083	2	4	6	8	10

NATURAL COSINES

[Numbers in difference columns to be subtracted, not added]

Degrees	0′ 0°·0	6′ 0°·1	12′ 0°·2	18′ 0°·3	24′ 0°·4	30′ 0°·5	36′ 0°·6	42′ 0°·7	48′ 0°·8	54′ 0°·9	Mean Differences 1	2	3	4	5
45	·7071	7059	7046	7034	7022	7009	6997	6984	6972	6959	2	4	6	8	10
46	·6947	6934	6921	6909	6896	6884	6871	6858	6845	6833	2	4	6	8	11
47	·6820	6807	6794	6782	6769	6756	6743	6730	6717	6704	2	4	6	9	11
48	·6691	6678	6665	6652	6639	6626	6613	6600	6587	6574	2	4	7	9	11
49	·6561	6547	6534	6521	6508	6494	6481	6468	6455	6441	2	4	7	9	11
50	·6428	6414	6401	6388	6374	6361	6347	6334	6320	6307	2	4	7	9	11
51	·6293	6280	6266	6252	6239	6225	6211	6198	6184	6170	2	5	7	9	11
52	·6157	6143	6129	6115	6101	6088	6074	6060	6046	6032	2	5	7	9	12
53	·6018	6004	5990	5976	5962	5948	5934	5920	5906	5892	2	5	7	9	12
54	·5878	5864	5850	5835	5821	5807	5793	5779	5764	5750	2	5	7	9	12
55	·5736	5721	5707	5693	5678	5664	5650	5635	5621	5606	2	5	7	10	12
56	·5592	5577	5563	5548	5534	5519	5505	5490	5476	5461	2	5	7	10	12
57	·5446	5432	5417	5402	5388	5373	5358	5344	5329	5314	2	5	7	10	12
58	·5299	5284	5270	5255	5240	5225	5210	5195	5180	5165	2	5	7	10	12
59	·5150	5135	5120	5105	5090	5075	5060	5045	5030	5015	3	5	8	10	13
60	·5000	4985	4970	4955	4939	4924	4909	4894	4879	4863	3	5	8	10	13
61	·4848	4833	4818	4802	4787	4772	4756	4741	4726	4710	3	5	8	10	13
62	·4695	4679	4664	4648	4633	4617	4602	4586	4571	4555	3	5	8	10	13
63	·4540	4524	4509	4493	4478	4462	4446	4431	4415	4399	3	5	8	10	13
64	·4384	4368	4352	4337	4321	4305	4289	4274	4258	4242	3	5	8	11	13
65	·4226	4210	4195	4179	4163	4147	4131	4115	4099	4083	3	5	8	11	13
66	·4067	4051	4035	4019	4003	3987	3971	3955	3939	3923	3	5	8	11	14
67	·3907	3891	3875	3859	3843	3827	3811	3795	3778	3762	3	5	8	11	14
68	·3746	3730	3714	3697	3681	3665	3649	3633	3616	3600	3	5	8	11	14
69	·3584	3567	3551	3535	3518	3502	3486	3469	3453	3437	3	5	8	11	14
70	·3420	3404	3387	3371	3355	3338	3322	3305	3289	3272	3	5	8	11	14
71	·3256	3239	3223	3206	3190	3173	3156	3140	3123	3107	3	6	8	11	14
72	·3090	3074	3057	3040	3024	3007	2990	2974	2957	2940	3	6	8	11	14
73	·2924	2907	2890	2874	2857	2840	2823	2807	2790	2773	3	6	8	11	14
74	·2756	2740	2723	2706	2689	2672	2656	2639	2622	2605	3	6	8	11	14
75	·2588	2571	2554	2538	2521	2504	2487	2470	2453	2436	3	6	8	11	14
76	·2419	2402	2385	2368	2351	2334	2317	2300	2284	2267	3	6	8	11	14
77	·2250	2233	2215	2198	2181	2164	2147	2130	2113	2096	3	6	9	11	14
78	·2079	2062	2045	2028	2011	1994	1977	1959	1942	1925	3	6	9	11	14
79	·1908	1891	1874	1857	1840	1822	1805	1788	1771	1754	3	6	9	11	14
80	·1736	1719	1702	1685	1668	1650	1633	1616	1599	1582	3	6	9	12	14
81	·1564	1547	1530	1513	1495	1478	1461	1444	1426	1409	3	6	9	12	14
82	·1392	1374	1357	1340	1323	1305	1288	1271	1253	1236	3	6	9	12	14
83	·1219	1201	1184	1167	1149	1132	1115	1097	1080	1063	3	6	9	12	14
84	·1045	1028	1011	0993	0976	0958	0941	0924	0906	0889	3	6	9	12	14
85	·0872	0854	0837	0819	0802	0785	0767	0750	0732	0715	3	6	9	12	15
86	·0698	0680	0663	0645	0628	0610	0593	0576	0558	0541	3	6	9	12	15
87	·0523	0506	0488	0471	0454	0436	0419	0401	0384	0366	3	6	9	12	15
88	·0349	0332	0314	0297	0279	0262	0244	0227	0209	0192	3	6	9	12	15
89	·0175	0157	0140	0122	0105	0087	0070	0052	0035	0017	3	6	9	12	15
90	·0000														

NATURAL TANGENTS

Degrees	0′ 0°·0	6′ 0°·1	12′ 0°·2	18′ 0°·3	24′ 0°·4	30′ 0°·5	36′ 0°·6	42′ 0°·7	48′ 0°·8	54′ 0°·9	Mean Differences 1	2	3	4	5
0	·0000	0017	0035	0052	0070	0087	0105	0122	0140	0157	3	6	9	12	15
1	·0175	0192	0209	0227	0244	0262	0279	0297	0314	0332	3	6	9	12	15
2	·0349	0367	0384	0402	0419	0437	0454	0472	0489	0507	3	6	9	12	15
3	·0524	0542	0559	0577	0594	0612	0629	0647	0664	0682	3	6	9	12	15
4	·0699	0717	0734	0752	0769	0787	0805	0822	0840	0857	3	6	9	12	15
5	·0875	0892	0910	0928	0945	0963	0981	0998	1016	1033	3	6	9	12	15
6	·1051	1069	1086	1104	1122	1139	1157	1175	1192	1210	3	6	9	12	15
7	·1228	1246	1263	1281	1299	1317	1334	1352	1370	1388	3	6	9	12	15
8	·1405	1423	1441	1459	1477	1495	1512	1530	1548	1566	3	6	9	12	15
9	·1584	1602	1620	1638	1655	1673	1691	1709	1727	1745	3	6	9	12	15
10	·1763	1781	1799	1817	1835	1853	1871	1890	1908	1926	3	6	9	12	15
11	·1944	1962	1980	1998	2016	2035	2053	2071	2089	2107	3	6	9	12	15
12	·2126	2144	2162	2180	2199	2217	2235	2254	2272	2290	3	6	9	12	15
13	·2309	2327	2345	2364	2382	2401	2419	2438	2456	2475	3	6	9	12	15
14	·2493	2512	2530	2549	2568	2586	2605	2623	2642	2661	3	6	9	12	16
15	·2679	2698	2717	2736	2754	2773	2792	2811	2830	2849	3	6	9	13	16
16	·2867	2886	2905	2924	2943	2962	2981	3000	3019	3038	3	6	9	13	16
17	·3057	3076	3096	3115	3134	3153	3172	3191	3211	3230	3	6	10	13	16
18	·3249	3269	3288	3307	3327	3346	3365	3385	3404	3424	3	6	10	13	16
19	·3443	3463	3482	3502	3522	3541	3561	3581	3600	3620	3	7	10	13	16
20	·3640	3659	3679	3699	3719	3739	3759	3779	3799	3819	3	7	10	13	17
21	·3839	3859	3879	3899	3919	3939	3959	3979	4000	4020	3	7	10	13	17
22	·4040	4061	4081	4101	4122	4142	4163	4183	4204	4224	3	7	10	14	17
23	·4245	4265	4286	4307	4327	4348	4369	4390	4411	4431	3	7	10	14	17
24	·4452	4473	4494	4515	4536	4557	4578	4599	4621	4642	4	7	11	14	18
25	·4663	4684	4706	4727	4748	4770	4791	4813	4834	4856	4	7	11	14	18
26	·4877	4899	4921	4942	4964	4986	5008	5029	5051	5073	4	7	11	15	18
27	·5095	5117	5139	5161	5184	5206	5228	5250	5272	5295	4	7	11	15	18
28	·5317	5340	5362	5384	5407	5430	5452	5475	5498	5520	4	8	11	15	19
29	·5543	5566	5589	5612	5635	5658	5681	5704	5727	5750	4	8	12	15	19
30	·5774	5797	5820	5844	5867	5890	5914	5938	5961	5985	4	8	12	16	20
31	·6009	6032	6056	6080	6104	6128	6152	6176	6200	6224	4	8	12	16	20
32	·6249	6273	6297	6322	6346	6371	6395	6420	6445	6469	4	8	12	16	20
33	·6494	6519	6544	6569	6594	6619	6644	6669	6694	6720	4	8	13	17	21
34	·6745	6771	6796	6822	6847	6873	6899	6924	6950	6976	4	9	13	17	21
35	·7002	7028	7054	7080	7107	7133	7159	7186	7212	7239	4	9	13	18	22
36	·7265	7292	7319	7346	7373	7400	7427	7454	7481	7508	5	9	14	18	23
37	·7536	7563	7590	7618	7646	7673	7701	7729	7757	7785	5	9	14	18	23
38	·7813	7841	7869	7898	7926	7954	7983	8012	8040	8069	5	9	14	19	24
39	·8098	8127	8156	8185	8214	8243	8273	8302	8332	8361	5	10	15	20	24
40	·8391	8421	8451	8481	8511	8541	8571	8601	8632	8662	5	10	15	20	25
41	·8693	8724	8754	8785	8816	8847	8878	8910	8941	8972	5	10	16	21	26
42	·9004	9036	9067	9099	9131	9163	9195	9228	9260	9293	5	11	16	21	27
43	·9325	9358	9391	9424	9457	9490	9523	9556	9590	9623	6	11	17	22	28
44	·9657	9691	9725	9759	9793	9827	9861	9896	9930	9965	6	11	17	23	29

NATURAL TANGENTS

Degrees	0′ 0°·0	6′ 0°·1	12′ 0°·2	18′ 0°·3	24′ 0°·4	30′ 0°·5	36′ 0°·6	42′ 0°·7	48′ 0°·8	54′ 0°·9	Mean Differences 1	2	3	4	5
45	1·0000	0035	0070	0105	0141	0176	0212	0247	0283	0319	6	12	18	24	30
46	1·0355	0392	0428	0464	0501	0538	0575	0612	0649	0686	6	12	18	25	31
47	1·0724	0761	0799	0837	0875	0913	0951	0990	1028	1067	6	13	19	25	32
48	1·1106	1145	1184	1224	1263	1303	1343	1383	1423	1463	7	13	20	27	33
49	1·1504	1544	1585	1626	1667	1708	1750	1792	1833	1875	7	14	21	28	34
50	1·1918	1960	2002	2045	2088	2131	2174	2218	2261	2305	7	14	22	29	36
51	1·2349	2393	2437	2482	2527	2572	2617	2662	2708	2753	8	15	23	30	38
52	1·2799	2846	2892	2938	2985	3032	3079	3127	3175	3222	8	16	24	31	39
53	1·3270	3319	3367	3416	3465	3514	3564	3613	3663	3713	8	16	25	33	41
54	1·3764	3814	3865	3916	3968	4019	4071	4124	4176	4229	9	17	26	34	43
55	1·4281	4335	4388	4442	4496	4550	4605	4659	4715	4770	9	18	27	36	45
56	1·4826	4882	4938	4994	5051	5108	5166	5224	5282	5340	10	19	29	38	48
57	1·5399	5458	5517	5577	5637	5697	5757	5818	5880	5941	10	20	30	40	50
58	1·6003	6066	6128	6191	6255	6319	6383	6447	6512	6577	11	21	32	43	53
59	1·6643	6709	6775	6842	6909	6977	7045	7113	7182	7251	11	23	34	45	56
60	1·7321	7391	7461	7532	7603	7675	7747	7820	7893	7966	12	24	36	48	60
61	1·8040	8115	8190	8265	8341	8418	8495	8572	8650	8728	13	26	38	51	64
62	1·8807	8887	8967	9047	9128	9210	9292	9375	9458	9542	14	27	41	55	68
63	1·9626	9711	9797	9883	9970	2·0057	2·0145	2·0233	2·0323	2·0413	15	29	44	58	73
64	2·0503	0594	0686	0778	0872	0965	1060	1155	1251	1348	16	31	47	63	78
65	2·1445	1543	1642	1742	1842	1943	2045	2148	2251	2355	17	34	51	68	85
66	2·2460	2566	2673	2781	2889	2998	3109	3220	3332	3445	18	37	55	73	92
67	2·3559	3673	3789	3906	4023	4142	4262	4383	4504	4627	20	40	60	79	99
68	2·4751	4876	5002	5129	5257	5386	5517	5649	5782	5916	22	43	65	87	108
69	2·6051	6187	6325	6464	6605	6746	6889	7034	7179	7326	24	47	71	95	119
70	2·7475	7625	7776	7929	8083	8239	8397	8556	8716	8878	26	52	78	104	131
71	2·9042	9208	9375	9544	9714	9887	3·0061	3·0237	3·0415	3·0595	29	58	87	116	145
72	3·0777	0961	1146	1334	1524	1716	1910	2106	2305	2506	32	64	96	129	161
73	3·2709	2914	3122	3332	3544	3759	3977	4197	4420	4646	36	72	108	144	180
74	3·4874	5105	5339	5576	5816	6059	6305	6554	6806	7062	41	81	122	163	204
75	3·7321	7583	7848	8118	8391	8667	8947	9232	9520	9812	46	93	139	186	232
76	4·0108	0408	0713	1022	1335	1653	1976	2303	2635	2972	53	107	160	213	267
77	4·3315	3662	4015	4374	4737	5107	5483	5864	6252	6646					
78	4·7046	7453	7867	8288	8716	9152	9594	5·0045	5·0504	5·0970	Mean differences cease to be sufficiently accurate.				
79	5·1446	1929	2422	2924	3435	3955	4486	5026	5578	6140					
80	5·6713	7297	7894	8502	9124	9758	6·0405	6·1066	6·1742	6·2432					
81	6·3138	3859	4596	5350	6122	6912	7720	8548	9395	7·0264					
82	7·1154	2066	3002	3962	4947	5958	6996	8062	9158	8·0285					
83	8·1443	2636	3863	5126	6427	7769	9152	9·0579	9·2052	9·3572					
84	9·5144	9·677	9·845	10·02	10·20	10·39	10·58	10·78	10·99	11·20					
85	11·43	11·66	11·91	12·16	12·43	12·71	13·00	13·30	13·62	13·95					
86	14·30	14·67	15·06	15·46	15·89	16·35	16·83	17·34	17·89	18·46					
87	19·08	19·74	20·45	21·20	22·02	22·90	23·86	24·90	26·03	27·27					
88	28·64	30·14	31·82	33·69	35·80	38·19	40·92	44·07	47·74	52·08					
89	57·29	63·66	71·62	81·85	95·49	114·6	143·2	191·0	286·5	573·0					
90	∞														

LOGARITHMS OF SINES

Degrees	0′ 0°·0	6′ 0°·1	12′ 0°·2	18′ 0°·3	24′ 0°·4	30′ 0°·5	36′ 0°·6	42′ 0°·7	48′ 0°·8	54′ 0°·9	Mean Differences 1	2	3	4	5
0	∞	$\bar{3}$·2419	$\bar{3}$·5429	7190	8439	9408	$\bar{2}$·0200	$\bar{2}$·0870	$\bar{2}$·1450	$\bar{2}$·1961					
1	$\bar{2}$·2419	2832	3210	3558	3880	4179	4459	4723	4971	5206					
2	$\bar{2}$·5428	5640	5842	6035	6220	6397	6567	6731	6889	7041					
3	$\bar{2}$·7188	7330	7468	7602	7731	7857	7979	8098	8213	8326					
4	$\bar{2}$·8436	8543	8647	8749	8849	8946	9042	9135	9226	9315	16	32	48	64	80
5	$\bar{2}$·9403	9489	9573	9655	9736	9816	9894	9970	$\bar{1}$·0046	$\bar{1}$·0120	13	26	39	52	65
6	$\bar{1}$·0192	0264	0334	0403	0472	0539	0605	0670	0734	0797	11	22	33	44	55
7	$\bar{1}$·0859	0920	0981	1040	1099	1157	1214	1271	1326	1381	10	19	29	38	48
8	$\bar{1}$·1436	1489	1542	1594	1646	1697	1747	1797	1847	1895	8	17	25	34	42
9	$\bar{1}$·1943	1991	2038	2085	2131	2176	2221	2266	2310	2353	8	15	23	30	38
10	$\bar{1}$·2397	2439	2482	2524	2565	2606	2647	2687	2727	2767	7	14	20	27	34
11	$\bar{1}$·2806	2845	2883	2921	2959	2997	3034	3070	3107	3143	6	12	19	25	31
12	$\bar{1}$·3179	3214	3250	3284	3319	3353	3387	3421	3455	3488	6	11	17	23	28
13	$\bar{1}$·3521	3554	3586	3618	3650	3682	3713	3745	3775	3806	5	11	16	21	26
14	$\bar{1}$·3837	3867	3897	3927	3957	3986	4015	4044	4073	4102	5	10	15	20	24
15	$\bar{1}$·4130	4158	4186	4214	4242	4269	4296	4323	4350	4377	5	9	14	18	23
16	$\bar{1}$·4403	4430	4456	4482	4508	4533	4559	4584	4609	4634	4	9	13	17	21
17	$\bar{1}$·4659	4684	4709	4733	4757	4781	4805	4829	4853	4876	4	8	12	16	20
18	$\bar{1}$·4900	4923	4946	4969	4992	5015	5037	5060	5082	5104	4	8	11	15	19
19	$\bar{1}$·5126	5148	5170	5192	5213	5235	5256	5278	5299	5320	4	7	11	14	18
20	$\bar{1}$·5341	5361	5382	5402	5423	5443	5463	5484	5504	5523	3	7	10	14	17
21	$\bar{1}$·5543	5563	5583	5602	5621	5641	5660	5679	5698	5717	3	6	10	13	16
22	$\bar{1}$·5736	5754	5773	5792	5810	5828	5847	5865	5883	5901	3	6	9	12	15
23	$\bar{1}$·5919	5937	5954	5972	5990	6007	6024	6042	6059	6076	3	6	9	12	15
24	$\bar{1}$·6093	6110	6127	6144	6161	6177	6194	6210	6227	6243	3	6	8	11	14
25	$\bar{1}$·6259	6276	6292	6308	6324	6340	6356	6371	6387	6403	3	5	8	11	13
26	$\bar{1}$·6418	6434	6449	6465	6480	6495	6510	6526	6541	6556	3	5	8	10	13
27	$\bar{1}$·6570	6585	6600	6615	6629	6644	6659	6673	6687	6702	2	5	7	10	12
28	$\bar{1}$·6716	6730	6744	6759	6773	6787	6801	6814	6828	6842	2	5	7	9	12
29	$\bar{1}$·6856	6869	6883	6896	6910	6923	6937	6950	6963	6977	2	4	7	9	11
30	$\bar{1}$·6990	7003	7016	7029	7042	7055	7068	7080	7093	7106	2	4	6	9	11
31	$\bar{1}$·7118	7131	7144	7156	7168	7181	7193	7205	7218	7230	2	4	6	8	10
32	$\bar{1}$·7242	7254	7266	7278	7290	7302	7314	7326	7338	7349	2	4	6	8	10
33	$\bar{1}$·7361	7373	7384	7396	7407	7419	7430	7442	7453	7464	2	4	6	8	10
34	$\bar{1}$·7476	7487	7498	7509	7520	7531	7542	7553	7564	7575	2	4	6	7	9
35	$\bar{1}$·7586	7597	7607	7618	7629	7640	7650	7661	7671	7682	2	4	5	7	9
36	$\bar{1}$·7692	7703	7713	7723	7734	7744	7754	7764	7774	7785	2	3	5	7	9
37	$\bar{1}$·7795	7805	7815	7825	7835	7844	7854	7864	7874	7884	2	3	5	7	8
38	[illegible]	7903	7913	7922	7932	7941	7951	7960	7970	7979	2	3	5	6	8
39	[illegible]	7998	8007	8017	8026	8035	8044	8053	8063	8072	2	3	5	6	8
40	[illegible]	8090	8099	8108	8117	8125	8134	8143	8152	8161	1	3	4	6	7
41	[illegible]	8178	8187	8195	8204	8213	8221	8230	8238	8247	1	3	4	6	7
42	[illegible]	8264	8272	8280	8289	8297	8305	8313	8322	8330	1	3	4	6	7
43	$\bar{1}$·8338	8346	8354	8362	8370	8378	8386	8394	8402	8410	1	3	4	5	7
44	$\bar{1}$·8418	8426	8433	8441	8449	8457	8464	8472	8480	8487	1	3	4	5	6

LOGARITHMS OF SINES

Degrees	0′ 0°·0	6′ 0°·1	12′ 0°·2	18′ 0°·3	24′ 0°·4	30′ 0°·5	36′ 0°·6	42′ 0°·7	48′ 0°·8	54′ 0°·9	Mean Differences 1	2	3	4	5
45	$\bar{1}$·8495	8502	8510	8517	8525	8532	8540	8547	8555	8562	1	2	4	5	6
46	$\bar{1}$·8569	8577	8584	8591	8598	8606	8613	8620	8627	8634	1	2	4	5	6
47	$\bar{1}$·8641	8648	8655	8662	8669	8676	8683	8690	8697	8704	1	2	3	5	6
48	$\bar{1}$·8711	8718	8724	8731	8738	8745	8751	8758	8765	8771	1	2	3	4	6
49	$\bar{1}$·8778	8784	8791	8797	8804	8810	8817	8823	8830	8836	1	2	3	4	5
50	$\bar{1}$·8843	8849	8855	8862	8868	8874	8880	8887	8893	8899	1	2	3	4	5
51	$\bar{1}$·8905	8911	8917	8923	8929	8935	8941	8947	8953	8959	1	2	3	4	5
52	$\bar{1}$·8965	8971	8977	8983	8989	8995	9000	9006	9012	9018	1	2	3	4	5
53	$\bar{1}$·9023	9029	9035	9041	9046	9052	9057	9063	9069	9074	1	2	3	4	5
54	$\bar{1}$·9080	9085	9091	9096	9101	9107	9112	9118	9123	9128	1	2	3	4	5
55	$\bar{1}$·9134	9139	9144	9149	9155	9160	9165	9170	9175	9181	1	2	3	3	4
56	$\bar{1}$·9186	9191	9196	9201	9206	9211	9216	9221	9226	9231	1	2	3	3	4
57	$\bar{1}$·9236	9241	9246	9251	9255	9260	9265	9270	9275	9279	1	2	2	3	4
58	$\bar{1}$·9284	9289	9294	9298	9303	9308	9312	9317	9322	9326	1	2	2	3	4
59	$\bar{1}$·9331	9335	9340	9344	9349	9353	9358	9362	9367	9371	1	1	2	3	4
60	$\bar{1}$·9375	9380	9384	9388	9393	9397	9401	9406	9410	9414	1	1	2	3	4
61	$\bar{1}$·9418	9422	9427	9431	9435	9439	9443	9447	9451	9455	1	1	2	3	3
62	$\bar{1}$·9459	9463	9467	9471	9475	9479	9483	9487	9491	9495	1	1	2	3	3
63	$\bar{1}$·9499	9503	9506	9510	9514	9518	9522	9525	9529	9533	1	1	2	3	3
64	$\bar{1}$·9537	9540	9544	9548	9551	9555	9558	9562	9566	9569	1	1	2	2	3
65	$\bar{1}$·9573	9576	9580	9583	9587	9590	9594	9597	9601	9604	1	1	2	2	3
66	$\bar{1}$·9607	9611	9614	9617	9621	9624	9627	9631	9634	9637	1	1	2	2	3
67	$\bar{1}$·9640	9643	9647	9650	9653	9656	9659	9662	9666	9669	1	1	2	2	3
68	$\bar{1}$·9672	9675	9678	9681	9684	9687	9690	9693	9696	9699	0	1	1	2	3
69	$\bar{1}$·9702	9704	9707	9710	9713	9716	9719	9722	9724	9727	0	1	1	2	2
70	$\bar{1}$·9730	9733	9735	9738	9741	9743	9746	9749	9751	9754	0	1	1	2	2
71	$\bar{1}$·9757	9759	9762	9764	9767	9770	9772	9775	9777	9780	0	1	1	2	2
72	$\bar{1}$·9782	9785	9787	9789	9792	9794	9797	9799	9801	9804	0	1	1	2	2
73	$\bar{1}$·9806	9808	9811	9813	9815	9817	9820	9822	9824	9826	0	1	1	2	2
74	$\bar{1}$·9828	9831	9833	9835	9837	9839	9841	9843	9845	9847	0	1	1	1	2
75	$\bar{1}$·9849	9851	9853	9855	9857	9859	9861	9863	9865	9867	0	1	1	1	2
76	$\bar{1}$·9869	9871	9873	9875	9876	9878	9880	9882	9884	9885	0	1	1	1	2
77	$\bar{1}$·9887	9889	9891	9892	9894	9896	9897	9899	9901	9902	0	1	1	1	1
78	$\bar{1}$·9904	9906	9907	9909	9910	9912	9913	9915	9916	9918	0	1	1	1	1
79	$\bar{1}$·9919	9921	9922	9924	9925	9927	9928	9929	9931	9932	0	0	1	1	1
80	$\bar{1}$·9934	9935	9936	9937	9939	9940	9941	9943	9944	9945	0	0	1	1	1
81	$\bar{1}$·9946	9947	9949	9950	9951	9952	9953	9954	9955	9956	0	0	1	1	1
82	$\bar{1}$·9958	9959	9960	9961	9962	9963	9964	9965	9966	9967	0	0	1	1	1
83	$\bar{1}$·9968	9968	9969	9970	9971	9972	9973	9974	9975	9975	0	0	0	1	1
84	$\bar{1}$·9976	9977	9978	9978	9979	9980	9981	9981	9982	9983	0	0	0	0	1
85	$\bar{1}$·9983	9984	9985	9985	9986	9987	9987	9988	9988	9989	0	0	0	0	0
86	$\bar{1}$·9989	9990	9990	9991	9991	9992	9992	9993	9993	9994	0	0	0	0	0
87	$\bar{1}$·9994	9994	9995	9995	9996	9996	9996	9996	9997	9997	0	0	0	0	0
88	$\bar{1}$·9997	9998	9998	9998	9998	9999	9999	9999	9999	9999	0	0	0	0	0
89	$\bar{1}$·9999	9999	0·0000	0000	0000	0000	0000	0000	0000	0000					
90	0·0000														

LOGARITHMS OF COSINES

[Numbers in difference columns to be subtracted, not added]

Degrees	0′ 0°·0	6′ 0°·1	12′ 0°·2	18′ 0°·3	24′ 0°·4	30′ 0°·5	36′ 0°·6	42′ 0°·7	48′ 0°·8	54′ 0°·9	Mean Differences 1	2	3	4	5
0	0·0000	0000	0000	0000	0000	0000	0000	0000	0000	$\bar{1}$·9999	0	0	0	0	0
1	$\bar{1}$·9999	9999	9999	9999	9999	9999	9998	9998	9998	9998	0	0	0	0	0
2	$\bar{1}$·9997	9997	9997	9996	9996	9996	9996	9995	9995	9994	0	0	0	0	0
3	$\bar{1}$·9994	9994	9993	9993	9992	9992	9991	9991	9990	9990	0	0	0	0	0
4	$\bar{1}$·9989	9989	9988	9988	9987	9987	9986	9985	9985	9984	0	0	0	0	0
5	$\bar{1}$·9983	9983	9982	9981	9981	9980	9979	9978	9978	9977	0	0	0	0	1
6	$\bar{1}$·9976	9975	9975	9974	9973	9972	9971	9970	9969	9968	0	0	0	1	1
7	$\bar{1}$·9968	9967	9966	9965	9964	9963	9962	9961	9960	9959	0	0	1	1	1
8	$\bar{1}$·9958	9956	9955	9954	9953	9952	9951	9950	9949	9947	0	0	1	1	1
9	$\bar{1}$·9946	9945	9944	9943	9941	9940	9939	9937	9936	9935	0	0	1	1	1
10	$\bar{1}$·9934	9932	9931	9929	9928	9927	9925	9924	9922	9921	0	0	1	1	1
11	$\bar{1}$·9919	9918	9916	9915	9913	9912	9910	9909	9907	9906	0	1	1	1	1
12	$\bar{1}$·9904	9902	9901	9899	9897	9896	9894	9892	9891	9889	0	1	1	1	1
13	$\bar{1}$·9887	9885	9884	9882	9880	9878	9876	9875	9873	9871	0	1	1	1	2
14	$\bar{1}$·9869	9867	9865	9863	9861	9859	9857	9855	9853	9851	0	1	1	1	2
15	$\bar{1}$·9849	9847	9845	9843	9841	9839	9837	9835	9833	9831	0	1	1	1	2
16	$\bar{1}$·9828	9826	9824	9822	9820	9817	9815	9813	9811	9808	0	1	1	2	2
17	$\bar{1}$·9806	9804	9801	9799	9797	9794	9792	9789	9787	9785	0	1	1	2	2
18	$\bar{1}$·9782	9780	9777	9775	9772	9770	9767	9764	9762	9759	0	1	1	2	2
19	$\bar{1}$·9757	9754	9751	9749	9746	9743	9741	9738	9735	9733	0	1	1	2	2
20	$\bar{1}$·9730	9727	9724	9722	9719	9716	9713	9710	9707	9704	0	1	1	2	2
21	$\bar{1}$·9702	9699	9696	9693	9690	9687	9684	9681	9678	9675	0	1	1	2	2
22	$\bar{1}$·9672	9669	9666	9662	9659	9656	9653	9650	9647	9643	1	1	2	2	3
23	$\bar{1}$·9640	9637	9634	9631	9627	9624	9621	9617	9614	9611	1	1	2	2	3
24	$\bar{1}$·9607	9604	9601	9597	9594	9590	9587	9583	9580	9576	1	1	2	2	3
25	$\bar{1}$·9573	9569	9566	9562	9558	9555	9551	9548	9544	9540	1	1	2	2	3
26	$\bar{1}$·9537	9533	9529	9525	9522	9518	9514	9510	9506	9503	1	1	2	3	3
27	$\bar{1}$·9499	9495	9491	9487	9483	9479	9475	9471	9467	9463	1	1	2	3	3
28	$\bar{1}$·9459	9455	9451	9447	9443	9439	9435	9431	9427	9422	1	1	2	3	3
29	$\bar{1}$·9418	9414	9410	9406	9401	9397	9393	9388	9384	9380	1	1	2	3	4
30	$\bar{1}$·9375	9371	9367	9362	9358	9353	9349	9344	9340	9335	1	1	2	3	4
31	$\bar{1}$·9331	9326	9322	9317	9312	9308	9303	9298	294	9289	1	2	2	3	4
32	$\bar{1}$·9284	9279	9275	9270	9265	9260	9255	9251	9246	9241	1	2	2	3	4
33	$\bar{1}$·9236	9231	9226	9221	9216	9211	9206	9201	9196	9191	1	2	3	3	4
34	$\bar{1}$·9186	9181	9175	9170	9165	9160	9155	9149	9144	9139	1	2	3	3	4
35	$\bar{1}$·9134	9128	9123	9118	9112	9107	9101	9096	9091	9085	1	2	3	4	5
36	$\bar{1}$·9080	9074	9069	9063	9057	9052	9046	9041	9035	9029	1	2	3	4	5
37	$\bar{1}$·9023	9018	9012	9006	9000	8995	8989	8983	8977	8971	1	2	3	4	5
38	$\bar{1}$·8965	8959	8953	8947	8941	8935	8929	8923	8917	8911	1	2	3	4	5
39	$\bar{1}$·8905	8899	8893	8887	8880	8874	8868	8862	8855	8849	1	2	3	4	5
40	$\bar{1}$·8843	8836	8830	8823	8817	8810	8804	8797	8791	8784	1	2	3	4	5
41	$\bar{1}$·8778	8771	8765	8758	8751	8745	8738	8731	8724	8718	1	2	3	5	6
42	$\bar{1}$·8711	8704	8697	8690	8683	8676	8669	8662	8655	8648	1	2	3	5	6
43	$\bar{1}$·8641	8634	8627	8620	8613	8606	8598	8591	8584	8577	1	2	4	5	6
44	$\bar{1}$·8569	8562	8555	8547	8540	8532	8525	8517	8510	8502	1	2	4	5	6

LOGARITHMS OF COSINES

[*Numbers in difference columns to be subtracted, not added.*]

Degrees	0′ 0°·0	6′ 0°·1	12′ 0°·2	18′ 0°·3	24′ 0°·4	30′ 0°·5	36′ 0°·6	42′ 0°·7	48′ 0°·8	54′ 0°·9	Mean Differences 1	2	3	4	5
45	$\bar{1}$·8495	8487	8480	8472	8464	8457	8449	8441	8433	8426	1	3	4	5	6
46	$\bar{1}$·8418	8410	8402	8394	8386	8378	8370	8362	8354	8346	1	3	4	5	7
47	$\bar{1}$·8338	8330	8322	8313	8305	8297	8289	8280	8272	8264	1	3	4	6	7
48	$\bar{1}$·8255	8247	8238	8230	8221	8213	8204	8195	8187	8178	1	3	4	6	7
49	$\bar{1}$·8169	8161	8152	8143	8134	8125	8117	8108	8099	8090	1	3	4	6	7
50	$\bar{1}$·8081	8072	8063	8053	8044	8035	8026	8017	8007	7998	2	3	5	6	8
51	$\bar{1}$·7989	7979	7970	7960	7951	7941	7932	7922	7913	7903	2	3	5	6	8
52	$\bar{1}$·7893	7884	7874	7864	7854	7844	7835	7825	7815	7805	2	3	5	7	8
53	$\bar{1}$·7795	7785	7774	7764	7754	7744	7734	7723	7713	7703	2	3	5	7	9
54	$\bar{1}$·7692	2682	7671	7661	7650	7640	7629	7618	7607	7597	3	4	5	7	9
55	$\bar{1}$·7586	7575	7564	7553	7542	7531	7520	7509	7498	7487	2	4	6	7	9
56	$\bar{1}$·7476	7464	7453	7442	7430	7419	7407	7396	7384	7373	2	4	6	8	10
57	$\bar{1}$·7361	7349	7338	7326	7314	7302	7290	7278	7266	7254	2	4	6	8	10
58	$\bar{1}$·7242	7230	7218	7205	7193	7181	7168	7156	7144	7131	2	4	6	8	10
59	$\bar{1}$·7118	7106	7093	7080	7068	7055	7042	7029	7016	7003	2	4	6	9	11
60	$\bar{1}$·6990	6977	6963	6950	6937	6923	6910	6896	6883	6869	2	4	7	9	11
61	$\bar{1}$·6856	6842	6828	6814	6801	6787	6773	6759	6744	6730	2	5	7	9	12
62	$\bar{1}$·6716	6702	6687	6673	6659	6644	6629	6615	6600	6585	2	5	7	10	12
63	$\bar{1}$·6570	6556	6541	6526	6510	6495	6480	6465	6449	6434	3	5	8	10	13
64	$\bar{1}$·6418	6403	6387	6371	6356	6340	6324	6308	6292	6276	3	5	8	11	13
65	$\bar{1}$·6259	6243	6227	6210	6194	6177	6161	6144	6127	6110	3	6	8	11	14
66	$\bar{1}$·6093	6076	6059	6042	6024	6007	5990	5972	5954	5937	3	6	9	12	15
67	$\bar{1}$·5919	5901	5883	5865	5847	5828	5810	5792	5773	5754	3	6	9	12	15
68	$\bar{1}$·5736	5717	5698	5679	5660	5641	5621	5602	5583	5563	3	6	10	13	16
69	$\bar{1}$·5543	5523	5504	5484	5463	5443	5423	5402	5382	5361	3	7	10	14	17
70	$\bar{1}$·5341	5320	5299	5278	5256	5235	5213	5192	5170	5148	4	7	11	14	18
71	$\bar{1}$·5126	5104	5082	5060	5037	5015	4992	4969	4946	4923	4	8	11	15	19
72	$\bar{1}$·4900	4876	4853	4829	4805	4781	4757	4733	4709	4684	4	8	12	16	20
73	$\bar{1}$·4659	4634	4609	4584	4559	4533	4508	4482	4456	4430	4	9	13	17	21
74	$\bar{1}$·4403	4377	4350	4323	4296	4269	4242	4214	4186	4158	5	9	14	18	23
75	$\bar{1}$·4130	4102	4073	4044	4015	3986	3957	3927	3897	3867	5	10	15	20	24
76	$\bar{1}$·3837	3806	3775	3745	3713	3682	3650	3618	3586	3554	5	11	16	21	26
77	$\bar{1}$·3521	3488	3455	3421	3387	3353	3319	3284	3250	3214	6	11	17	23	28
78	$\bar{1}$·3179	3143	3107	3070	3034	2997	2959	2921	2883	2845	6	12	19	25	31
79	$\bar{1}$·2806	2767	2727	2687	2647	2606	2565	2524	2482	2439	7	14	20	27	34
80	$\bar{1}$·2397	2353	2210	2266	2221	2176	2131	2085	2038	1991	8	15	23	30	38
81	$\bar{1}$·1943	1895	1847	1797	1747	1697	1646	1594	1542	1489	8	17	25	34	42
82	$\bar{1}$·1436	1381	1326	1271	1214	1157	1099	1040	0981	0920	10	19	29	38	48
83	$\bar{1}$·0859	0797	0734	0670	0605	0539	0472	0403	0334	0264	11	22	33	44	55
84	$\bar{1}$·0192	0120	0046	$\bar{2}$·9970	$\bar{2}$·9894	$\bar{2}$·9816	$\bar{2}$·9736	$\bar{2}$·9655	$\bar{2}$·9573	$\bar{2}$·9489	13	26	39	52	65
85	$\bar{2}$·9403	9315	9226	9135	9042	8946	8849	8749	8647	8543	16	32	48	64	80
86	$\bar{2}$·8436	8326	8213	8098	7979	7857	7731	7602	7468	7330					
87	$\bar{2}$·7188	7041	6889	6731	6567	6397	6220	6035	5842	5640					
88	$\bar{2}$·5428	5206	4971	4723	4459	4179	3880	3558	3210	2832					
89	$\bar{2}$·2419	1961	1450	0870	0200	$\bar{3}$·9408	$\bar{3}$·8439	$\bar{3}$·7190	$\bar{3}$·5429	$\bar{3}$·2419					
90	∞														

LOGARITHMS OF TENGENTS

Degrees	0′ 0°·0	6′ 0°·1	12′ 0°·2	18′ 0°·3	24′ 0°·4	30′ 0°·5	36′ 0°·6	42′ 0°·7	48′ 0°·8	54′ 0°·9	Mean Differences 1	2	3	4	5
0	$-\infty$	$\bar{3}$·2419	$\bar{3}$·5429	$\bar{3}$·7190	$\bar{3}$·8439	$\bar{3}$·9409	$\bar{2}$·0200	$\bar{2}$·0870	$\bar{2}$·1450	$\bar{2}$·1962					
1	$\bar{2}$·2419	2833	3211	3559	3881	4181	4461	4725	4973	5208					
2	$\bar{2}$·5431	5643	5845	6038	6223	6401	6571	6736	6894	7046					
3	$\bar{2}$·7194	7337	7475	7609	7739	7865	7988	8107	8223	8336					
4	$\bar{2}$·8446	8554	8659	8762	8862	8960	9056	9150	9241	9331	16	32	48	64	81
5	$\bar{2}$·9420	9506	9591	9674	9756	9836	9915	9992	$\bar{1}$·0068	$\bar{1}$·0143	13	26	40	53	66
6	$\bar{1}$·0216	0289	0360	0430	0499	0567	0633	0699	0764	0828	11	22	34	45	56
7	$\bar{1}$·0891	0954	1015	1076	1135	1194	1252	1310	1367	1423	10	20	29	39	49
8	$\bar{1}$·1478	1533	1587	1640	1693	1745	1797	1848	1898	1948	9	17	26	35	43
9	$\bar{1}$·1997	2046	2094	2142	2189	2236	2282	2328	2374	2419	8	16	23	31	39
10	$\bar{1}$·2463	2507	2551	2594	2637	2680	2722	2764	2805	2846	7	14	21	28	35
11	$\bar{1}$·2887	2927	2967	3006	3046	3085	3123	3162	3200	3237	6	13	19	26	32
12	$\bar{1}$·3275	3312	3349	3385	3422	3458	3493	3529	3564	3599	6	12	18	24	30
13	$\bar{1}$·3634	3668	3702	3736	3770	3804	3837	3870	3903	3935	6	11	17	22	28
14	$\bar{1}$·3968	4000	4032	4064	4095	4127	4158	4189	4220	4250	5	10	16	21	26
15	$\bar{1}$·4281	4311	4341	4371	4400	4430	4459	4488	4517	4546	5	10	15	20	25
16	$\bar{1}$·4575	4603	4632	4660	4688	4716	4744	4771	4799	4826	5	9	14	19	23
17	$\bar{1}$·4853	4880	4907	4934	4961	4987	5014	5040	5066	5092	4	9	13	18	22
18	$\bar{1}$·5118	5143	5169	5195	5220	5245	5270	5295	5320	5345	4	8	13	17	21
19	$\bar{1}$·5370	5394	5419	5443	5467	5491	5516	5539	5563	5587	4	8	12	16	20
20	$\bar{1}$·5611	5634	5658	5681	5704	5727	5750	5773	5796	5819	4	8	12	15	19
21	$\bar{1}$·5842	5864	5887	5909	5932	5954	5976	5998	6020	6042	4	7	11	15	19
22	$\bar{1}$·6064	6086	6108	6129	6151	6172	6194	6215	6236	6257	4	7	11	14	18
23	$\bar{1}$·6279	6300	6321	6341	6362	6383	6404	6424	6445	6465	3	7	10	14	17
24	$\bar{1}$·6486	6506	6527	6547	6567	6587	6607	6627	6647	6667	3	7	10	13	17
25	$\bar{1}$·6687	6706	6726	6746	6765	6785	6804	6824	6843	6863	3	7	10	13	16
26	$\bar{1}$·6882	6901	6920	6939	6958	6977	6996	7015	7034	7053	3	6	9	13	16
27	$\bar{1}$·7072	7090	7109	7128	7146	7165	7183	7202	7220	7238	3	6	9	12	15
28	$\bar{1}$·7257	7275	7293	7311	7330	7348	7366	7384	7402	7420	3	6	9	12	15
29	$\bar{1}$·7438	7455	7473	7491	7509	7526	7544	7562	7579	7597	3	6	9	12	15
30	$\bar{1}$·7614	7632	7649	7667	7684	7701	7719	7736	7753	7771	3	6	9	12	14
31	$\bar{1}$·7788	7805	7822	7839	7856	7873	7890	7907	7924	7941	3	6	9	11	14
32	$\bar{1}$·7958	7975	7992	8008	8025	8042	8059	8075	8092	8109	3	6	8	11	14
33	$\bar{1}$·8125	8142	8158	8175	8191	8208	8224	8241	8257	8274	3	5	8	11	14
34	$\bar{1}$·8290	8306	8323	8339	8355	8371	8388	8404	8420	8436	3	5	8	11	14
35	$\bar{1}$·8452	8468	8484	8501	8517	8533	8549	8565	8581	8597	3	5	8	11	13
36	$\bar{1}$·8613	8629	8644	8660	8676	8692	8708	8724	8740	8755	3	5	8	11	13
37	$\bar{1}$·8771	8787	8803	8818	8834	8850	8865	8881	8897	8912	3	5	8	10	13
38	$\bar{1}$·8928	8944	8959	8975	8990	9006	9022	9037	9053	9068	3	5	8	10	13
39	$\bar{1}$·9084	9099	9115	9130	9146	9161	9176	9192	9207	9223	3	5	8	10	13
40	$\bar{1}$·9238	9254	9269	9284	9300	9315	9330	9346	9361	9376	3	5	8	10	13
41	$\bar{1}$·9392	9407	9422	9438	9453	9468	9483	9499	9514	9529	3	5	8	10	13
42	$\bar{1}$·9544	9560	9575	9590	9605	9621	9636	9651	9666	9681	3	5	8	10	13
43	$\bar{1}$·9697	9712	9727	9742	9757	9772	9788	9803	9818	9833	3	5	8	10	13
44	$\bar{1}$·9848	9864	9879	9894	9909	9924	9939	9955	9970	9985	3	5	8	10	13